Mathematical modeling and practicals on perfect acoustic absorption by Helmholtz resonators

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IUMPA Instituto Universitario de Matemática Pura y Aplicada

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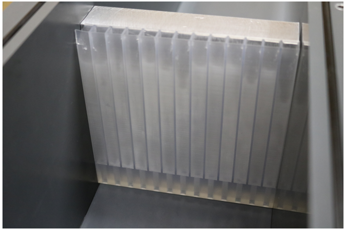

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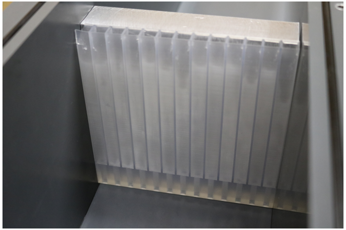

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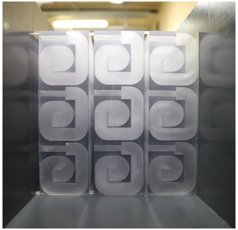

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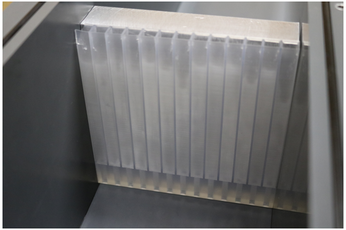

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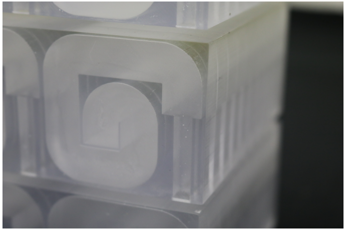

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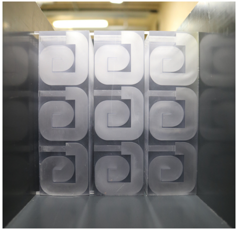

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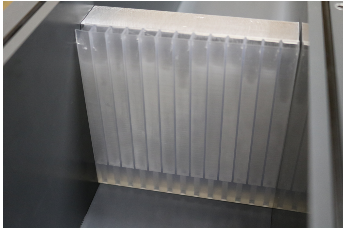

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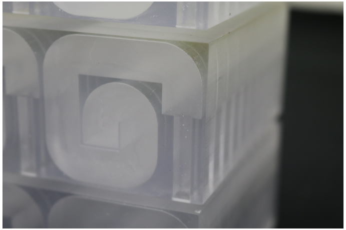

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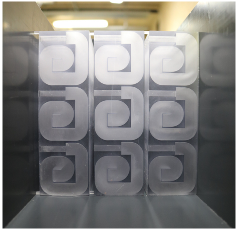

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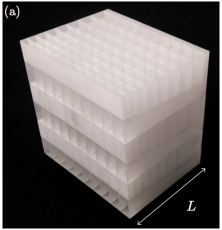

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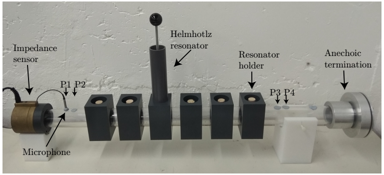
1. Modeling of the analyzed system

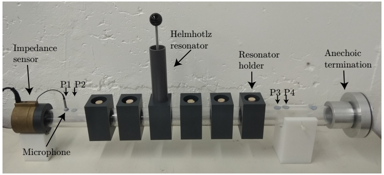
Hypothesis



What we consider is:

- Acoustic wave propagation in cylindrical waveguides.
- Viscous and thermal effects: dissipative fluid and boundary layers.

Hypothesis



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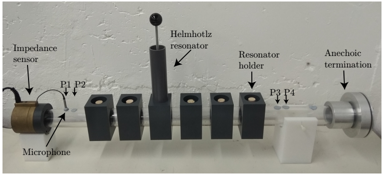
Hypothesis

- Wave propagation in isotropic and homogeneous fluids.
- Plane wave propagation, 1D reciprocal problem.
- Punctual resonators.

Hypothesis



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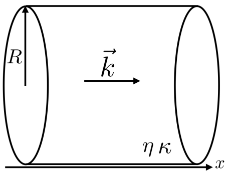
What we use:

- Transfer matrix method.
- Effective parameters to consider viscous and thermal effects.
- Correction lengths in the resonator elements.

Wave propagation in cylindrical tubes



$$\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} + \frac{1}{r^2}\frac{\partial^2 p}{\partial \theta^2} + \frac{\partial^2 p}{\partial x^2} + k^2 p = 0,$$

$$k^2 = k_r^2 + k_x^2,$$

$$p(r,\theta,x,\omega) = R(r)\Theta(\theta)X(x)e^{-i\omega t},$$

$$p(r,\theta,x) = [A_r J_m(k_r r) + B_r N_m(k_r r)]\left[A_\theta e^{-im\theta} + B_\theta e^{im\theta}\right]\left[A_x e^{-ik_x x} + B_x e^{ik_x x}\right].$$

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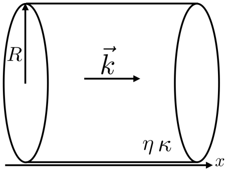

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- Geometric conditions, $B_r = 0$.
- Neumann type boundary conditions at $r = R$, $\frac{\partial p}{\partial r}|_{r=R} = 0$, $J'(k_r R) = 0$.
- Propagating waves along x^+, $A_x = 0$.

Cut-off frequency

- $J'(k_r R) = 0 \rightarrow k_r^{mn} R = j^{mn}$.
- First mode, m,n=1,0.
- $f_c = 1.84 \frac{c}{2\pi R}$.

Propagation of plane waves in cylindrical narrow tubes

Consider a tube of radius R containing an ideal gas of viscosity η and thermal conductivity κ.

$$\rho_0 \frac{\partial \vec{v}}{\partial t} = -\vec{\nabla} p + \frac{4}{3}\eta \vec{\nabla}(\vec{\nabla}\vec{v}) - \eta \vec{\nabla} \times \vec{\nabla} \times \vec{v},$$

$$\frac{\partial \rho}{\partial t} = -\rho_0 \vec{\nabla}\vec{v},$$

$$\kappa \vec{\nabla} T = \frac{T_0}{P_0}\left(\rho_0 C_v \frac{\partial p}{\partial t} - P_0 C_p \frac{\partial \rho}{\partial t}\right),$$

$$\frac{\partial p}{\partial t} = \frac{P_0}{\rho_0 T_0}\left(\rho_0 \frac{\partial T}{\partial t} + T_0 \frac{\partial \rho}{\partial t}\right).$$

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Lossy case $e^{-i\omega t}$

$$-i\omega \rho_0 \vec{v} = -\vec{\nabla} p + \frac{4}{3}\eta \vec{\nabla}(\vec{\nabla}\vec{v}) - \eta \vec{\nabla} \times \vec{\nabla} \times \vec{v},$$
$$i\omega \rho = \rho_0 \vec{\nabla} \vec{v},$$
$$\kappa \vec{\nabla} T = -\frac{i\omega T_0}{P_0}\left(\rho_0 C_v p - P_0 C_p \rho\right),$$
$$p = \frac{P_0}{\rho_0 T_0}\left(\rho_0 T + T_0 \rho\right).$$

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$$p = \frac{P_0}{\rho_0 T_0}\left(\rho_0 T + T_0 \rho\right).$$

Lossless case $e^{-i\omega t}$

$$i\omega\rho_0\vec{v} = \vec{\nabla} p,$$
$$i\omega\rho = \rho_0 \vec{\nabla}\vec{v},$$
$$\rho_0 p = P_0 \gamma \rho, \quad \gamma \equiv C_p/C_v,$$
$$p = \frac{P_0}{\rho_0 T_0}\left(\rho_0 T + T_0 \rho\right).$$

Propagation of plane waves in cylindrical narrow tubes

Considering that $R > 10^{-3}$ cm and the waveguide is filled with air, the average velocity and the excess density are given by [1]

$$i\omega\rho(\omega) < u >= \frac{dp}{dx}, \qquad B(\omega) = \frac{\rho_0 p}{< \rho >},$$

Effective mass density

$$\rho = \rho_0 \left(1 - \frac{2J_1(R\tilde{G}_r)}{R\tilde{G}_r J_0(R\tilde{G}_r)} \right)^{-1},$$

$$\tilde{G}_r = \sqrt{i\omega\rho_0/\eta}.$$

Effective bulk modulus

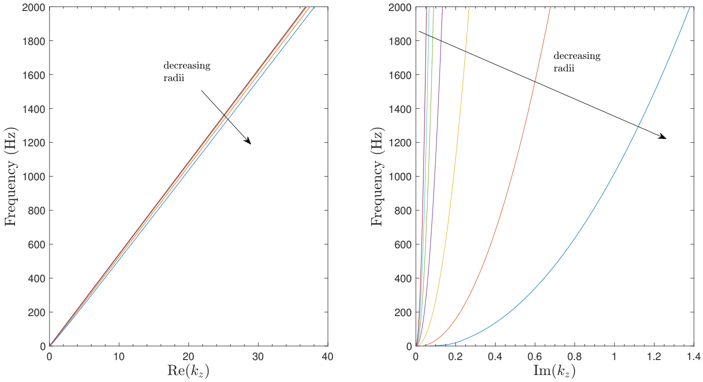
$$B = \gamma P_0 \left(1 + (\gamma - 1)\frac{2J_1(R\tilde{G}_k)}{R\tilde{G}_k J_0(R\tilde{G}_k)} \right)^{-1},$$

$\tilde{G}_k = \sqrt{i\omega \, \mathrm{Pr} \, \rho_0/\eta}$, Pr Prandtl number

The effects of viscosity and thermal conduction are well separated.

[1] G. Kirchhoff, Ann. Phys. Chem., 134, 177 (1868); C. Zwikker and C.W. Kosten, Sound Absorbing Materials (Elsevier, Amsterdam, 1949); H. Tijdeman, J. Sound. Vic., 39, 1 (1975); D.E. Weston, Proc. Phys. Soc. London Sec. B, 66, 95, (1953); M.R. Stinson, J. Acoust. Soc. Am., 89, 550, (1991).

Propagation constant: $k_x = \pm\sqrt{\omega^2 \rho(\omega)/B(\omega)}$.
$R = [0.001 \;\; 0.002 \;\; 0.005 \;\; 0.01 \;\; 0.015 \;\; 0.02 \;\; 0.025]$ m.

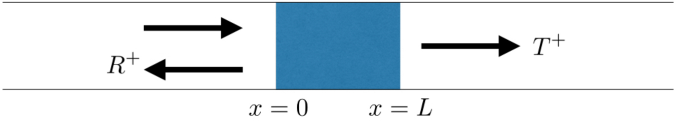


- Hypothesis
  - 1D propagation, i.e., plane wave propagation.
  - Propagation in homogeneous and isotropic materials.
- Definition
  - The transfer matrix between the two faces, $x = 0$ and $x = L$, of 1D material, T, is used to relate the sound pressure, P, and normal acoustic particle velocity, V,

$$\begin{bmatrix} P \\ V \end{bmatrix}_{x=0} = \mathrm{T} \begin{bmatrix} P \\ V \end{bmatrix}_{x=L} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} P \\ V \end{bmatrix}_{x=L}, \qquad (1)$$

Transmission problem

We start by solving the problem with a left hand side incident wave.



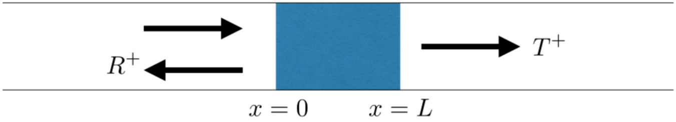
When the incident plane wave is assumed of unitary amplitude, the sound pressures and particle velocities on the two surfaces of the layer become

$$P|_{x=0} = 1 + R^+, \quad (2)$$
$$V|_{x=0} = \frac{1}{Z_0}(1 - R^+), \quad (3)$$

$$P|_{x=L} = T^+ e^{ikL}, \quad (4)$$
$$V|_{x=L} = \frac{T^+ e^{ikL}}{Z_0}. \quad (5)$$

Transmission problem

We start by solving the problem of incidence from the left side.



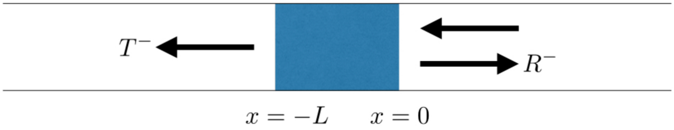
We can represent the reflection and transmission coefficients with a left hand side incident wave as:

$$T^+ = \frac{2e^{-ikL}}{T_{11} + T_{12}/Z_0 + Z_0 T_{21} + T_{22}}, \quad (6)$$

$$R^+ = \frac{T_{11} + T_{12}/Z_0 - Z_0 T_{21} - T_{22}}{T_{11} + T_{12}/Z_0 + Z_0 T_{21} + T_{22}}. \quad (7)$$

Transmission problem

Now we solve the problem with a right hand side incidence wave.



We can represent the reflection and transmission coefficients with a right hand side incident wave:

$$T^- = \frac{2e^{-ikL}(T_{11}T_{22} - T_{12}T_{21})}{T_{11} + T_{12}/Z_0 + Z_0 T_{21} + T_{22}}, \quad (8)$$

$$R^- = \frac{-T_{11} + T_{12}/Z_0 - Z_0 T_{21} + T_{22}}{T_{11} + T_{12}/Z_0 + Z_0 T_{21} + T_{22}}. \quad (9)$$

Transmission problem

- If the system is symmetric, then, $R^+ = R^-$, and as a consequence, $T_{11} = T_{22}$.
- For reciprocal systems, $T^+ = T^-$, then the transfer matrix is unitary ($T_{11}T_{22} - T_{12}T_{21} = 1$). This property is satisfied for linear and time invariant systems.

The reflection and transmission coefficients are

$$T^- = \frac{2e^{ikL}(T_{11}T_{22} - T_{12}T_{21})}{T_{11} + T_{12}/Z_0 + Z_0T_{21} + T_{22}}, \quad (10)$$

$$R^- = \frac{-T_{11} + T_{12}/Z_0 - Z_0T_{21} + T_{22}}{T_{11} + T_{12}/Z_0 + Z_0T_{21} + T_{22}}, \quad (11)$$

$$R^+ = \frac{T_{11} + T_{12}/Z_0 - Z_0T_{21} - T_{22}}{T_{11} + T_{12}/Z_0 + Z_0T_{21} + T_{22}}. \quad (12)$$

Absorption in Transmission problem

- For symmetric systems the absorption coefficient is the same whatever the incidence side, and is calculated as

$$\alpha = 1 - |R|^2 - |T|^2. \tag{13}$$

- For asymmetric systems the absorption depends on the side of the incidence. For the positive x-axis incoming waves, the asymmetric absorption is given by

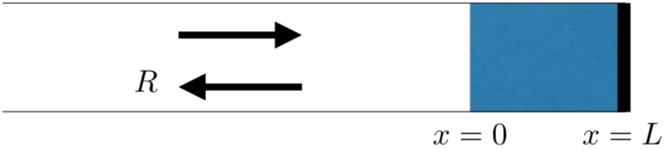
$$\alpha^- = 1 - |R^-|^2 - |T|^2, \tag{14}$$

while for the negative x-axis incoming waves,

$$\alpha^+ = 1 - |R^+|^2 - |T|^2. \tag{15}$$

Reflection problem

Now, we consider that the L-thick layer of material is positioned against a rigid backing.



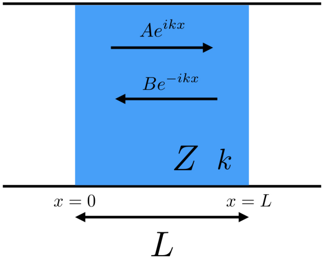
On the wall, $V_{x=L} = 0$. Then, the reflection coefficient is,

$$R = \frac{T_{11} - Z_0 T_{21}}{T_{11} + Z_0 T_{21}}, \tag{16}$$

and, finally, the absorption coefficient of the rigidly-backed system reduces to

$$\alpha = 1 - |R|^2. \tag{17}$$

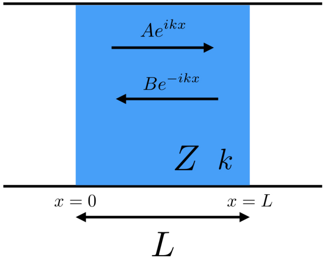
Homogeneous isotropic material



- $P(x) = Ae^{ikx} + Be^{-ikx}$,
- $V(x) = \frac{A}{Z}e^{ikx} - \frac{B}{Z}e^{-ikx}$.

where Z is the impedance and k the wavenumber.

Homogeneous isotropic material



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where Z is the impedance and k the wavenumber.

- $P(0) = A + B$,
- $V(0) = \frac{A}{Z} - \frac{B}{Z}$.
- $P(L) = (A + B)\cos(kL) + i(A - B)Z\sin(kL)$,
- $V(L) = \frac{(A+B)}{Z}\cos(kL) + \frac{i(A-B)}{Z}\sin(kL)$,

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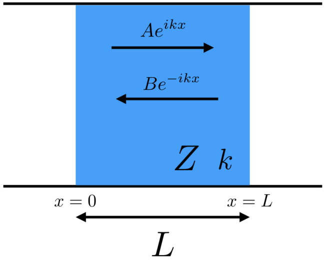

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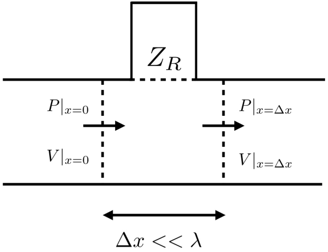
Homogeneous isotropic material

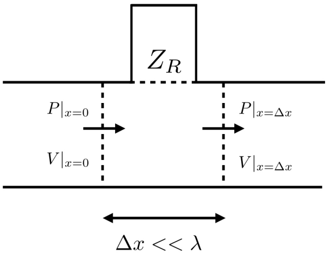
$$\begin{bmatrix} \cos(kL) & -iZ\sin(kL) \\ \frac{-i}{Z}\sin(kL) & \cos(kL) \end{bmatrix}.$$

Parallel resonant branch (continuity of pressure)



- We consider a punctual resonator such that $\Delta x << \lambda$.
- The pressures just before and just after the resonator are the same as the pressure at the opening of the resonator.

Parallel resonant branch (continuity of pressure)



The presence of the side branch resonator produces a flow drop:

$$Z_R = \frac{P|_{\Delta x}}{-\Delta V} = \frac{P|_{\Delta x}}{V|_0 - V|_{\Delta x}}.$$

$P(0) = P(\Delta x),$
$V(0) = V(\Delta x) + Z_R V(0).$

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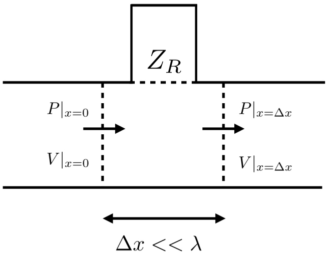

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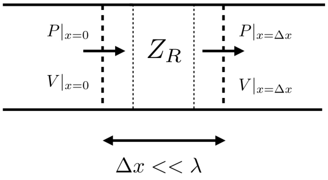
- We consider a punctual resonator such that $\Delta x << \lambda$.
- The pressures just before and just after the resonator are the same as the pressure at the opening of the resonator.

Parallel resonant branch

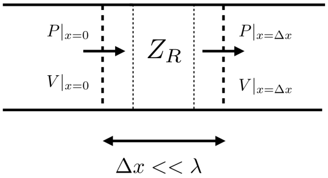
$$\begin{bmatrix} 1 & 0 \\ \frac{1}{Z_R} & 1 \end{bmatrix}.$$

Series impedance branch (continuity of particle velocity)



- We consider punctual resonator as $\Delta x << \lambda$.
- The particle velocities just before and just after the resonator are the same as the particle velocity of the resonator.

Series impedance branch (continuity of particle velocity)



- We consider punctual resonator as $\Delta x << \lambda$.
- The particle velocities just before and just after the resonator are the same as the particle velocity of the resonator.

The presence of the resonator in series produces a pressure drop:

$$Z_R = \frac{\Delta P}{V} = \frac{P|_0 - P|_{\Delta x}}{V|_{\Delta x}}.$$

$$V(0) = V(\Delta x),$$
$$P(0) = P(\Delta x) + Z_R V(\Delta x).$$

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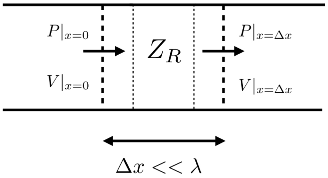

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Series impedance branch

$$\begin{bmatrix} 1 & Z_R \\ 0 & 1 \end{bmatrix}$$

Flux formulation

When the fluid is confined in waveguides of different cross-sectional area, reflections are produced at the discontinuities. In this case, it is convinient reformulate the problem in terms of the flux, $\mathcal{V}$.

$$\mathcal{V} = SV$$

where S is the cross-sectional area of the waveguide. In this cas, what we have is

$$\left[\begin{array}{c} P \\ \mathcal{V}/S \end{array}\right]_{x=0} = \left[\begin{array}{cc} T_{11} & T_{12} \\ T_{21} & T_{22} \end{array}\right] \left[\begin{array}{c} P \\ \mathcal{V}/S \end{array}\right]_{x=L}. \tag{18}$$

$$\left[\begin{array}{c} P \\ \mathcal{V} \end{array}\right]_{x=0} = \left[\begin{array}{cc} T_{11} & T_{12}/S \\ T_{21}/S & T_{22} \end{array}\right] \left[\begin{array}{c} P \\ \mathcal{V} \end{array}\right]_{x=L}. \tag{19}$$

Summary

In this notes, we are using $e^{-i\omega t}$.

- Transfer matrix of a slab of thickness L made of an isotropic material characterized by a wavenumber k and the characteristic impedance $Z' = Z/S$ is,

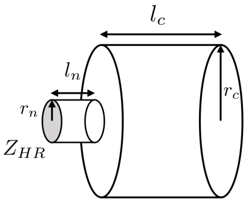
$$\begin{bmatrix} P \\ \mathcal{V} \end{bmatrix}_{x=0} = \begin{bmatrix} \cos(kL) & -iZ'\sin(kL) \\ -\frac{i}{Z'}\sin(kL) & \cos(kL) \end{bmatrix} \begin{bmatrix} P \\ \mathcal{V} \end{bmatrix}_{x=L}. \tag{20}$$

- Parallel impedance branch (continuity of pressure).

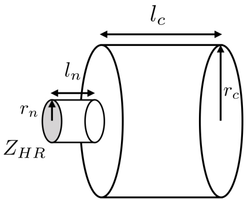
$$\begin{bmatrix} P \\ \mathcal{V} \end{bmatrix}_{x=0} = \begin{bmatrix} 1 & 0 \\ \frac{1}{Z'_R} & 1 \end{bmatrix} \begin{bmatrix} P \\ \mathcal{V} \end{bmatrix}_{x=L}. \tag{21}$$

- Series impedance branch (discontinuity of pressure).

$$\begin{bmatrix} P \\ \mathcal{V} \end{bmatrix}_{x=0} = \begin{bmatrix} 1 & Z'_R \\ 0 & 1 \end{bmatrix} \begin{bmatrix} P \\ \mathcal{V} \end{bmatrix}_{x=L}. \tag{22}$$



- Assembling of two cylindrical tubes of different cross section.
- Viscothermal losses accounted in the system via Stinson's formula.

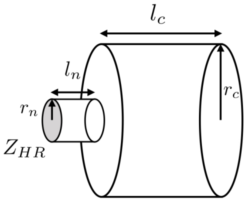


- Assembling of two cylindrical tubes of different cross section.
- Viscothermal losses accounted in the system via Stinson's formula.

$$\mathrm{T} = \begin{bmatrix} \cos(k_n l_n) & -iZ'_n \sin(k_n l_n) \\ -i\sin(k_n l_n)/Z'_n & \cos(k_n l_n) \end{bmatrix} \begin{bmatrix} \cos(k_c l_c) & -iZ'_c \sin(k_c l_c) \\ -i\sin(k_c l_c)/Z'_c & \cos(k_c l_c) \end{bmatrix}.$$

where $Z'_n = Z_n/S_n$ and $Z'_c = Z_c/S_c$. Applying the rigid boundary condition at the end of the Helmholtz resonator, we obtain

$$Z'_R = iZ'_n \frac{Z'_c/Z'_n - \tan k_n l_n \tan k_c l_c}{Z'_c/Z'_n \tan k_n l_n + \tan k_c l_c},$$



For low frequencies, when $k_n l_n \ll 1$ and $k_c l_c \ll 1$, then, $\tan(k_n l_n) \approx k_n l_n$ and $\tan(k_c l_n) \approx k_c l_c$. So we obtain

$$Z'_R = iZ'_n \frac{Z'_c/Z'_n - k_n l_n k_c l_c}{Z'_c/Z'_n k_n l_n + k_c l_c}.$$

If losses are not considered, $k_n = k_c = k_0$ and $Z_n = Z_c = Z_0$, with $k_0 = \omega/c_0$ and $Z_0 = \rho_0 c_0$. The first resonance of the HR is observed when $\Im\mathrm{m}(Z'_R) = 0$, we also obtain

$$\omega_R = c_0 \sqrt{\frac{S_n}{l_n l_c S_c}} = c_0 \sqrt{\frac{S_n}{l_n V_c}},$$

which is the usual expression for the resonance frequency of a Helmholtz resonator, where $V_c = S_c l_c$ is the volume of the cavity.

End corrections

The radiation at discontinuities must be included.

$$T = \begin{bmatrix} \cos(k_n \Delta l_1) & -iZ'_n \sin(k_n \Delta l_1) \\ -i\sin(k_n \Delta l_1)/Z'_n & \cos(k_n \Delta l_1) \end{bmatrix} \begin{bmatrix} \cos(k_n l_n) & -iZ'_n \sin(k_n l_n) \\ -i\sin(k_n l_n)/Z'_n & \cos(k_n l_n) \end{bmatrix}$$
$$\begin{bmatrix} \cos(k_n \Delta l_2) & -iZ'_n \sin(k_n \Delta l_2) \\ -i\sin(k_n \Delta l_2)/Z'_n & \cos(k_n \Delta l_2) \end{bmatrix} \begin{bmatrix} \cos(k_c l_c) & -iZ'_c \sin(k_c l_c) \\ -i\sin(k_c l_c)/Z'_c & \cos(k_c l_c) \end{bmatrix}.$$

Δl_1, is due to pressure radiation at the discontinuity from the neck duct to the cavity of the Helmholtz resonator . Δl_2 comes from the radiation at the discontinuity from the neck to the principal waveguide.
Considering the low frequency approximation, we get

$$Z'_R = iZ'_n \frac{\cos k_n l_n \cos k_c l_c - \frac{k_n \Delta l Z'_n}{Z'_c} \cos k_n l_n \sin k_c l_c - \frac{Z'_n}{Z'_c} \sin k_n l_n \sin k_c l_c}{\sin k_n l_n \cos k_c l_c - \frac{k_n \Delta l Z'_n}{Z'_c} \sin k_n l_n \sin k_c l_c + \frac{Z'_n}{Z'_c} \cos k_n l_n \sin k_c l_c},$$

Change of section in a waveguide. The length correction, Δl, is approximated by [2],

$$\Delta l = 0.82\left[1 - 1.35\frac{r_n}{r_c} + 0.31\left(\frac{r_n}{r_c}\right)^3\right] r_n.$$

where r_n is the radius of the small waveguide, e.g., the neck, and r_c is the radius of the big waveguide, e.g, the cavity of a cylindrical Helmholtz resonator.

A tube is loaded in parallel to the principal waveguide. The length of the end correction Δl is given by [3]

$$\Delta l = 0.82\left[1 - 0.235\frac{r_n}{r_s} - 1.32\left(\frac{r_n}{r_t}\right)^2 + 1.54\left(\frac{r_n}{r_t}\right)^3 - 0.86\left(\frac{r_n}{r_t}\right)^4\right] r_n,$$

where r_n is the radius of the loading waveguide and r_t is the radius of the main waveguide.

[2] J. Kergomard and A. Garcia, J. Sound Vib.114, 465 (1987)

[3] V. Dubos *et al.* Acta Acustica united with Acustica, 85, 153 (1999).

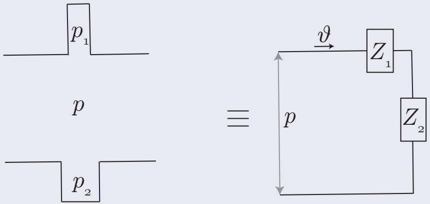
Exercise

Consider a cylindrical tube, with radius R and length L. The tube has a rigid termination at $x = L$ and it is open at $x = 0$. Obtain the characteristic impedance at $x = 0$.

This corresponds to the impedance of a quater wavelength resonator, can you justify this sentence?

Remember we are in the low frequency approximation (!)

Series circuit



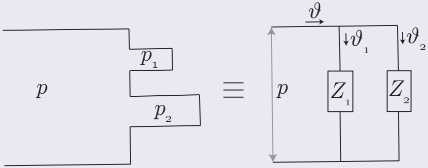
$$p = p_1 + p_2$$
$$\mathcal{V} = \mathcal{V}_1 = \mathcal{V}_2$$

Then,

$$\frac{p}{\mathcal{V}} = \frac{p_1}{\mathcal{V}_1} + \frac{p_2}{\mathcal{V}_1} = \frac{p_1}{\mathcal{V}_1} + \frac{p_2}{\mathcal{V}_2}$$
$$Z' = Z_1' + Z_2'$$

For n elements, $$\boxed{Z' = \sum_{i=1}^{n} Z_i'}$$

Parallel circuit



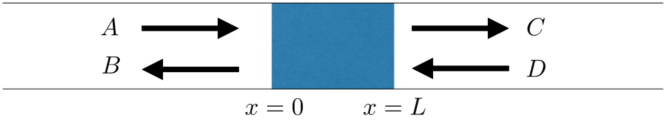
$$p = p_1 = p_2$$
$$\mathcal{V} = \mathcal{V}_1 + \mathcal{V}_2$$

Then,

$$\frac{p}{\mathcal{V}} = \frac{p_1}{\mathcal{V}_1 + \mathcal{V}_2} = \frac{1}{\frac{\mathcal{V}_1}{p_1} + \frac{\mathcal{V}_2}{p_2}}$$
$$Z'^{-1} = Z_1'^{-1} + Z_2'^{-1}$$

For n elements, $$\boxed{Z'^{-1} = \sum_{i=1}^{n} Z_i'^{-1}}$$

The scattering matrix, S, relates the amplitudes of the incoming waves to the system with those of the out-coming waves.



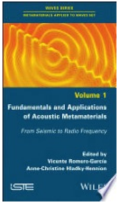
We consider that the total pressure in a point $x < 0$ ($x > L$) is given by $p(x) = Ae^{ikx} + Be^{-ikx}$ ($p(x) = Ce^{ikx} + De^{-ikx}$). Then,

$$\begin{bmatrix} C \\ B \end{bmatrix} = \mathrm{S} \begin{bmatrix} A \\ D \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} A \\ D \end{bmatrix} = \begin{bmatrix} T & R^{+} \\ R^{-} & T \end{bmatrix} \begin{bmatrix} A \\ D \end{bmatrix}. \tag{23}$$

The relation between T and S is then given by the Eqs.(6-14).

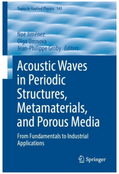
Exercise. Obtain the eigenvalues and eigenvectors of the S-matrix.

Reviews about perfect absorption for acoustic waves



- **Slow sound and critical coupling to design deep subwavelength acoustic metamaterials for perfect absorption and efficient diffusion**
  V. Romero-García, N. Jiménez and J.-P. Groby.
  In: V. Romero-García and A.-C. Hladky-Hennion (Eds.), Fundamentals and applications of acoustic metamaterials: from seismic to radiofrequency. ISTE Ltd. and John Wiley & Sons, Inc., London, UK, (2019)

- **Design of acoustic metamaterials made of Helmholtz resonators for perfect absorption by using the complex frequency plane**
  V. Romero-García, N. Jimenez, G. Theocharis, V. Achilleos, A. Merkel, O. Richoux, V. Tournat, J.-P. Groby and V. Pagneux.
  Comptes Rendus. Physique, 21, 7-8, 713-749, (2020).



- **Acoustic Metamaterial Absorbers**. J.-P. Groby, N. Jiménez, V. Romero-García.
  In: N. Jiménez, O. Umnova, J.-P. Groby. (Eds.), Acoustic Waves in Periodic Structures, Metamaterials, and Porous Media. Topics in Applied Physics, vol 143, pp. 167-204. Springer, (2021)

Mathematical modeling and practicals on perfect acoustic absorption by Helmholtz resonators

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