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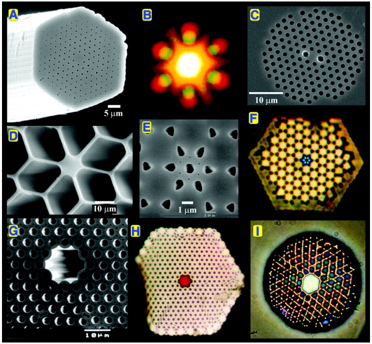
Structured media: Connecting the Microstructure to the Macroscale

Richard Craster Department of Mathematics, Imperial College London

Lectures in two pieces: first periodic media, then wave guides (then see how they connect). For periodic media:

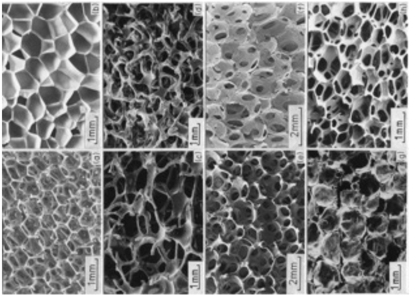
- Practical applications
- Homogenization theory: conventional theory valid for low frequencies and long waves. High frequency means multiple scattering and wavelength close to micro-scale. Need a new idea...
- Bloch waves: perfect lattices and quasi-periodicity.
- General theory: continuous media
- Elastic plates
- Rayleigh-Bloch/interface waves
- Examples from photonics: All angle negative refraction, ultra-refraction and localised defect modes.
- Concluding remarks.

Typical Structures: Photonic crystals (Optics)



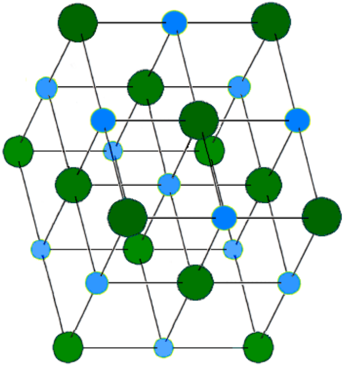
Micrographs of various Photonic Crystal Fibre structures taken from the review of Russell (Science 2003). The regular array of holes allow for excellent (low-loss) waveguides in optics and have a host of applications: sensors, high bandwidth guides, optical filters etc.

Engineering foams



Photographs of cellular solids: (a) open-cell polyurethane (b) closed-cell polyurethane (c) nickel (d) copper (e) zirconia (f) mullite (g) glass (h) a polyether foam with both open and closed cells. Taken from the book of Gibson and Ashby, 1997

Ranges of length scales



Discrete crystal atomic lattice structures in solid state physics and the Eiffel tower (made of periodic trusses and frames).

Continuous periodic composite structures: photonic and phononic crystals.
Frame structures: Lattice trusses, strings or beams creating a structure.
Discrete atom structures: Mass spring models - completely discrete.

All of these involve a well defined microscale (possibly many thousands, millions of cells) and a macroscale. Modelling waves through this is awkward, particularly when the wavelength and microscale is of the same order - would like to deal with macroscale equations with the microscale "built-in" through effective parameters.

Homogenization theory

A huge research area with many thousands of articles, numerous books. Almost all of this is either static or quasi-static: long wave and low frequency, so the wavelengths are much longer than the microscale. For instance, taking a piecewise constant elastic string on $-\infty < x < \infty$

$$l^2 \frac{d^2u}{dx^2} + \Omega^2 \frac{u}{c^2(\xi)} = 0, \qquad \text{with} \quad \Omega = \frac{\omega l}{\hat{c}_0}. \tag{0.1}$$

with $\xi = x/l$ and speed

$$c(\xi) = \begin{cases} 1/r \quad \text{for} \quad n \leq \xi < n+1 \\ 1 \quad \text{for} \quad n-1 \leq \xi < n \end{cases}$$

(n even).

A key idea is that there are two scales: $\xi = x/l$ a short scale and $X = x/L$ a long scale, with $\epsilon = l/L$. Note it is conventional to have a subtley different scaling $y = \xi/\epsilon$ and ξ, but I choose another. (Discuss) Then treat ξ, X as independent quantities so $u(x) = u(\xi, X)$ and

$$\partial_x = \frac{1}{l}(\partial_\xi + \epsilon \partial_X).$$

$$u_{\xi\xi} + 2\epsilon u_{\xi X} + \epsilon^2 u_{XX} + \frac{\Omega^2 u}{c^2(\xi)} = 0$$

Now if the frequency is low so $\Omega^2 \sim \epsilon^2 \Omega_2^2$ we expand

$$u(\xi, X) = u_0(\xi, X) + \epsilon u_1(\xi, X) + \epsilon^2 u_2(\xi, X) + ...$$

To leading order

$$u_{0\xi\xi} = 0$$

so $u_0(\xi, X) = u_0(X)$ (after noting implied periodicity in ξ).
Next order $u_1 = u_1(X)$ just absorb into u_0 and then finally

$$u_{2\xi\xi} = -(u_{0XX} + u_0 \frac{\Omega_2^2}{c^2(\xi)})$$

solvability means the RHS integrated from -1 to 1 wrt ξ is zero and hence

$$u_{0XX}(X) + u_0(X) \frac{\Omega_2^2}{2} \int_{-1}^{1} \frac{1}{c^2(\xi)} d\xi = 0.$$

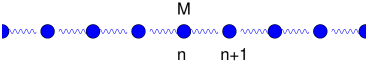
One just replaces the inverse speed squared effectively by its "effective" speed which is just the average. Naturally attractive approach giving equations only on the long scale and details built into an average.

Perfect periodic systems

Perfect systems are attractive, consider a chain of identical masses. If they are connected by simple springs with identical spring constant

$$y_{n+1} + y_{n-1} - 2y_n = -M\Omega^2 y_n$$

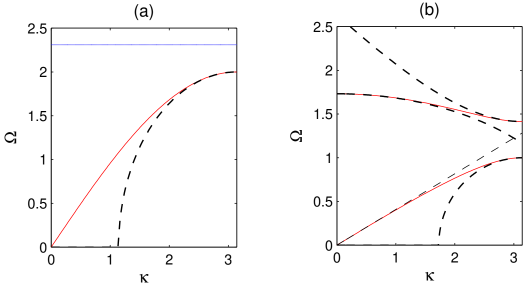
for integer n. Ω is the frequency.
A simple model



For a perfect lattice assume that $y_{n+1} = e^{i\kappa} y_n$ where κ is the phase shift. A key item of interest is the dispersion relation that connects the phase-shift to frequency

$$\Omega = \frac{2}{\sqrt{M}} \sin\left(\frac{\kappa}{2}\right)$$

Note it is linear for small Ω and $\Omega \sim \kappa/\sqrt{M}$ - so dispersionless at small frequencies and long waves.



Dispersion curves for the one-dimensional uniform lattice, (a) and the diatomic lattice (b). The exact dispersion curves are the solid lines whilst the asymptotics are the dashed lines. In panel (a) the dashed line above the exact curve shows the frequency associated to the localised defect state. In panel (a) the mass value $M = 1$ whilst in (b) $M_1 = 2$ and $M_2 = 1$.

Homogenization for simple chain

We begin by introducing a long-scale continuous variable $\eta = \epsilon n$ where ϵ is some small parameter The frequency is $\Omega = \epsilon\hat{\Omega}$ (where $\hat{\Omega}$) is an order one quantity. Let us set

$$y_n = y(\eta), \quad y_{n\pm 1} = y(\eta \pm \epsilon) \tag{0.2}$$

and then the difference equation becomes, in this new language, that

$$y(\eta + \epsilon) + y(\eta - \epsilon) - 2y(\eta) - M\epsilon^2\hat{\Omega}^2 y(\eta) \tag{0.3}$$

An expansion in a Taylor series

$$y(\eta + \epsilon) \sim y(\eta) + \epsilon y'(\eta) + \frac{\epsilon^2}{2} y''(\eta) + \dots \tag{0.4}$$

yields, to leading order,

$$y_{\eta\eta} + M\hat{\Omega}^2 y = 0. \tag{0.5}$$

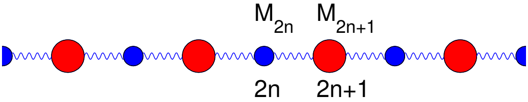
This is simply the wave equation for a string and suggests that if the wave was long enough that it would see the collection of masses as being smeared out to produce an effective string.
Notably the dispersion relation one obtains from the effective string is

$$\kappa = \sqrt{M}\Omega \tag{0.6}$$

when one replaces η with ϵn etc.

Bloch waves

Given a perfect infinite lattice in 1D or 2D can consider a single "cell". Classical example is the diatomic chain of masses and springs.



After non-dimensionalization the displacements y_{2n} satisfy

$$y_{2n-1} + y_{2n+1} - 2y_{2n} = -M_{2n}\Omega^2 y_{2n}$$

$$y_{2n} + y_{2n+2} - 2y_{2n+1} = -M_{2n+1}\Omega^2 y_{2n+1}.$$

Set $M_{2n} = M_2, M_{2n+1} = M_1$ then can just consider a cell y_{2n}, y_{2n+1} and use a vector notation $\mathbf{y}_{2n} = (y_{2n}, y_{2n+1})^T$.
Floquet-Bloch conditions are set across the cell

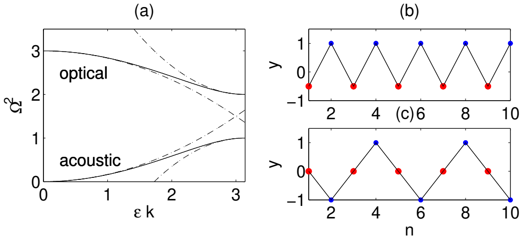
$$\mathbf{y}_{2n+2} = \exp(i\kappa)\mathbf{y}_{2n}.$$

Dispersion relation

The Bloch wavenumber κ plays a vital role - the phase shift across a cell - and is related to the frequency via a dispersion relation

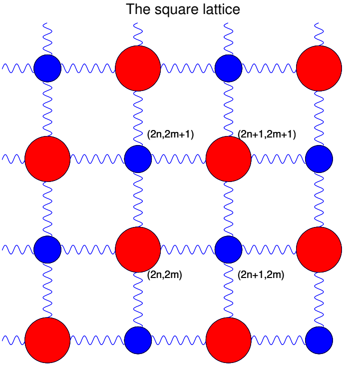
$$M_1 M_2 \Omega^4 - 2(M_1 + M_2)\Omega^2 + 2(1 - \cos\kappa) = 0.$$

Note range of κ and standing waves at end of Brillouin zone. Dashed lines from asymptotics (this is sufficiently simple that they can be verified analytically), really want to allow for varying masses, 2D, no longer perfect lattice, so not Bloch, find a continuum PDE for the high frequency vibration of the lattice etc etc. Before doing so more Bloch waves ...



2D lattice

Similar ideas hold for perfect lattices in higher dimensions - some useful extra details relevant for the general theory later. Consider, say, a square lattice of alternating masses:



$$y_{2n+1,2m} + y_{2n-1,2m} + y_{2n,2m+1} + y_{2n,2m-1} - 4y_{2n,2m} = -M_1\Omega^2 y_{2n,2m}$$

$$y_{2n+2,2m+1} + y_{2n,2m+1} + y_{2n+1,2m+2} + y_{2n+1,2m} - 4y_{2n+1,2m+1}$$
$$= -M_1\Omega^2 y_{2n+1,2m+1}$$

$$y_{2n+2,2m} + y_{2n,2m} + y_{2n+1,2m+1} + y_{2n+1,2m-1} - 4y_{2n+1,2m} = -M_2\Omega^2 y_{2n+1,2m}$$

$$y_{2n+1,2m+1} + y_{2n-1,2m+1} + y_{2n,2m+2} + y_{2n,2m} - 4y_{2n,2m+1} = -M_2\Omega^2 y_{2n,2m+1}$$

2D dispersion relation

Consider an infinite medium (just use a cell of four masses) and utilise the Bloch relation,

$$\mathbf{y}_{2n+\hat{N},2m+\hat{M}} = \exp(i[\kappa_1\hat{N} + \kappa_2\hat{M}])\mathbf{y}_{2n,2m} \tag{0.7}$$

with Bloch wavenumber vector $\kappa = (\kappa_1, \kappa_2)$. The exact dispersion relation is the eigenvalue problem

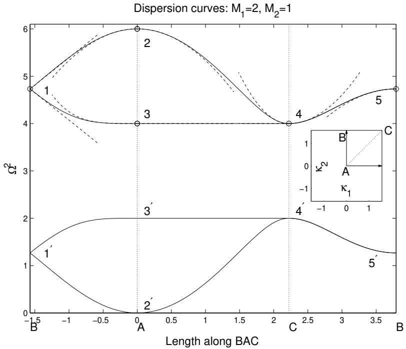
$$[A(\kappa) - \Omega^2 M]\mathbf{y}_{2n,2m} = 0. \tag{0.8}$$

Here $\mathbf{y}_{2n,2m}$ is the displacement vector

$$\mathbf{y}_{2n,2m} = [y_{2n,2m}, y_{2n+1,2m+1}, y_{2n+1,2m}, y_{2n,2m+1}]^T,$$

$M = \mathrm{diag}[M_1, M_1, M_2, M_2]$, and $A(\kappa)$ is the Hermitian matrix

$$A(\kappa) = \begin{pmatrix} 4 & 0 & -(1+e^{-2i\kappa_1}) & -(1+e^{-2i\kappa_2}) \\ 0 & 4 & -(1+e^{2i\kappa_2}) & -(1+e^{2i\kappa_1}) \\ -(1+e^{2i\kappa_1}) & -(1+e^{-2i\kappa_2}) & 4 & 0 \\ -(1+e^{2i\kappa_2}) & -(1+e^{-2i\kappa_1}) & 0 & 4 \end{pmatrix}$$



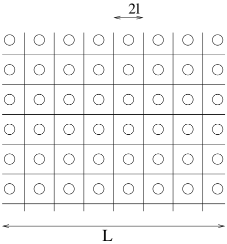
Dispersion curves for the square lattice: the inset shows shows the reciprocal lattice in $\kappa = (\kappa_1, \kappa_2)$ space, the irreducible Brillouin zone is completely characterised by the triangle ABC. The dotted lines in the upper half of the figure are asymptotic results.

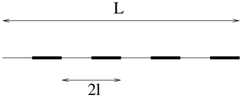
Note flat regions, positive/ negative group velocities, degenerate cases.

Also standing wave structure: periodic-periodic, periodic - antiperiodic, antiperiodic-antiperiodic.

General theory

Turn to a continuum with a double periodic microstructure - aim to generate a continuum description only on the macroscale. Then return to Bloch cases to verify model, then to non-periodic cases....





Consider a medium with two lengthscales L and l where $L \gg l$, set $\epsilon = l/L \ll 1$ for future use.

The microstructure is characterized by stiffness $\hat{a}(x_1/l, x_2/l)$ and density $\hat{\rho}(x_1/l, x_2/l)$ that are periodic on the microscale $\boldsymbol{\xi} = (x_1/l, x_2/l)$.

Consider a wave equation, for, say, SH waves in anti-plane elasticity with periodic density with time harmonic dependence $\exp(-i\omega t)$ assumed understood, as

$$l^2\nabla_{\mathbf{x}}\cdot[a(\boldsymbol{\xi})\nabla_{\mathbf{x}}u(\mathbf{x})]+\Omega^2\rho(\boldsymbol{\xi})u(\mathbf{x})=0 \quad \text{with} \quad \Omega=\frac{\omega l}{\hat{c}_0}$$

with $\hat{c}_0=\sqrt{\hat{a}_0/\hat{\rho}_0}$ a characteristic wave speed.
Adopt a multiple scales approach treating the disparate lengthscales $\mathbf{X}=\mathbf{x}/L$, and $\boldsymbol{\xi}=\mathbf{x}/l$ as new independent variables to get

$$\nabla_{\boldsymbol{\xi}}\cdot[a(\boldsymbol{\xi})\nabla_{\boldsymbol{\xi}}u(\mathbf{X},\boldsymbol{\xi})]+\Omega^2\rho(\boldsymbol{\xi})u(\mathbf{X},\boldsymbol{\xi})$$
$$+\epsilon[2a(\boldsymbol{\xi})\nabla_{\boldsymbol{\xi}}+\nabla_{\boldsymbol{\xi}}a(\boldsymbol{\xi})]\cdot\nabla_{\mathbf{X}}u(\mathbf{X},\boldsymbol{\xi})+\epsilon^2a(\boldsymbol{\xi})\nabla_{\mathbf{X}}^2u(\mathbf{X},\boldsymbol{\xi})=0$$

As noted when looking at Bloch waves there are standing waves that are locally periodic-periodic on the microscale - then $u(\mathbf{X},\boldsymbol{\xi})$ periodic in $\boldsymbol{\xi}$, but not necessarily in $\mathbf{X}$.

$$u|_{\xi_1=1}=u|_{\xi_1=-1}, \qquad u|_{\xi_2=1}=u|_{\xi_2=-1},$$
$$u_{\xi_1}|_{\xi_1=1}=u_{\xi_1}|_{\xi_1=-1}, \qquad u_{\xi_2}|_{\xi_2=1}=u_{\xi_2}|_{\xi_2=-1}.$$

Asymptotic theory

Adopt the ansatz:

$$u(\mathbf{X}, \boldsymbol{\xi}) = u_0(\mathbf{X}, \boldsymbol{\xi}) + \epsilon u_1(\mathbf{X}, \boldsymbol{\xi}) + \epsilon^2 u_2(\mathbf{X}, \boldsymbol{\xi}) + \ldots, \quad \Omega^2 = \Omega_0^2 + \epsilon \Omega_1^2 + \epsilon^2 \Omega_2^2 + \ldots$$

Each $u_i(\mathbf{X}, \boldsymbol{\xi})$ for $i = 1, 2 \ldots$, is periodic in $\boldsymbol{\xi}$.
Importantly, this is not limited to $\Omega^2 \ll 1$ as in classical homogenization for which

$$u(\mathbf{X}, \boldsymbol{\xi}) = u_0(\mathbf{X}) + \ldots, \quad \Omega^2 = \epsilon^2 \Omega_2^2 + \ldots$$

Now solve order-by-order in ϵ.
At leading order

$$\nabla_{\boldsymbol{\xi}} \cdot [a(\boldsymbol{\xi}) \nabla_{\boldsymbol{\xi}} u_0] + \Omega_0^2 \rho(\boldsymbol{\xi}) u_0 = 0$$

A discrete spectrum of eigenvalues Ω_0^2 for which there is no phase shift across the structure and standing wave is formed. Solution is (simple eigenvalue)

$$u_0(\mathbf{X}, \boldsymbol{\xi}) = f_0(\mathbf{X}) U_0(\boldsymbol{\xi}, \Omega_0) \tag{0.9}$$

where $U_0(\boldsymbol{\xi}, \Omega_0)$ is a periodic function of $\boldsymbol{\xi}$, is known as is Ω_0. $f_0(\mathbf{X})$ is unknown and varies only on the macroscale.

First order

The equation for $u_1(\mathbf{X}, \boldsymbol{\xi})$ is

$$\nabla_{\boldsymbol{\xi}} \cdot [a(\boldsymbol{\xi}) \nabla_{\boldsymbol{\xi}} u_1] + \Omega_0^2 \rho(\boldsymbol{\xi}) u_1 = -\nabla_{\mathbf{X}} f_0 \cdot [2a(\boldsymbol{\xi}) \nabla_{\boldsymbol{\xi}} U_0 + U_0 \nabla_{\boldsymbol{\xi}} a(\boldsymbol{\xi})] - f_0 \Omega_1^2 \rho(\boldsymbol{\xi}) U_0$$

and we now invoke an orthogonality condition, integrating over a cell one finds that

$$\iint_S \left(U_0 \nabla_{\boldsymbol{\xi}} \cdot [a(\boldsymbol{\xi}) \nabla_{\boldsymbol{\xi}} u_1] + \Omega_0^2 \rho(\boldsymbol{\xi}) U_0 u_1 \right) dS$$
$$= -\nabla_{\mathbf{X}} f_0 \cdot \iint_S \nabla_{\boldsymbol{\xi}} [a(\boldsymbol{\xi}) U_0^2] dS - f_0 \Omega_1^2 \iint_S \rho(\boldsymbol{\xi}) U_0^2 dS$$

and further that

$$0 = \iint_S (U_0 \nabla_{\boldsymbol{\xi}} \cdot [a(\boldsymbol{\xi}) \nabla_{\boldsymbol{\xi}} u_1] - u_1 \nabla_{\boldsymbol{\xi}} \cdot [a(\boldsymbol{\xi}) \nabla_{\boldsymbol{\xi}} U_0]) dS = -f_0 \Omega_1^2 \iint_S \rho(\boldsymbol{\xi}) U_0^2 dS.$$

So Ω_1 must be identically zero.
An explicit solution for $u_1(\mathbf{X}, \boldsymbol{\xi})$ is

$$u_1(\mathbf{X}, \boldsymbol{\xi}) = f_1(\mathbf{X}) U_0(\boldsymbol{\xi}, \Omega_0) + \nabla_{\mathbf{X}} f_0(\mathbf{X}) \cdot [\mathbf{V}_1(\boldsymbol{\xi}, \Omega_0) - \boldsymbol{\xi} U_0(\boldsymbol{\xi}, \Omega_0)].$$

Auxiliary function

The vector function $\mathbf{V}_1 = (V_1^{(1)}, V_1^{(2)})$ satisfies

$$\nabla_{\boldsymbol{\xi}} \cdot [a(\boldsymbol{\xi})\nabla_{\boldsymbol{\xi}}]\mathbf{V}_1 + \Omega_0^2 \rho(\boldsymbol{\xi})\mathbf{V}_1 = 0, \tag{0.10}$$

but each component of $\mathbf{V}_1$ must be linearly independent of $U_0(\boldsymbol{\xi}, \Omega_0)$, choose it to be non-periodic. $u_1(\mathbf{X}, \boldsymbol{\xi})$ itself must be periodic in $\boldsymbol{\xi}$. Achieve this by selecting each individual component of $\mathbf{V}_1$ to be periodic along one of the ξ_i and then choose its boundary conditions along the other ξ_j, $j \neq i$, in such a way that condition u_1 itself is periodic. Do this on a single cell and then periodically continue to the full structure. $V_1^{(1)}(\boldsymbol{\xi}, \Omega_0)$ it taken to have periodicity in ξ_2 and then periodicity of u_1 in ξ_1 results in

$$V_1^{(1)}|_{\xi_1=1} - V_1^{(1)}|_{\xi_1=-1} = 2U_0|_{\xi_1=1} \tag{0.11}$$

$$V_{1\xi_1}^{(1)}|_{\xi_1=1} - V_{1\xi_1}^{(1)}|_{\xi_1=-1} = 2U_{0\xi_1}|_{\xi_1=1}. \tag{0.12}$$

Similar procedure for $V_1^{(2)}(\boldsymbol{\xi}, \Omega_0)$.

Continuum equation

$$\nabla_{\boldsymbol{\xi}} \cdot [a(\boldsymbol{\xi}) \nabla_{\boldsymbol{\xi}} u_2] + \Omega_0^2 \rho(\boldsymbol{\xi}) u_2 =$$
$$-a(\boldsymbol{\xi}) U_0 \nabla_{\mathbf{X}}^2 f_0 - [2a(\boldsymbol{\xi}) \nabla_{\boldsymbol{\xi}} + \nabla_{\boldsymbol{\xi}} a(\boldsymbol{\xi})] \cdot \nabla_{\mathbf{X}} u_1 - \Omega_2^2 \rho(\boldsymbol{\xi}) f_0 U_0$$

contains both $f_0(\mathbf{X})$ and the eigenvalue correction, Ω_2^2. Invoking an orthogonality condition, integrating over the cell yields an eigenvalue problem for f_0 and Ω_2^2 as the partial differential equation

$$T_{ij} \frac{\partial^2 f_0}{\partial X_i \partial X_j} + \Omega_2^2 f_0 = 0, \quad \text{with} \quad T_{ij} = \frac{t_{ij}}{\iint_S \rho(\boldsymbol{\xi}) U_0^2 dS} \quad \text{for} \quad i, j = 1, 2.$$

$$t_{11} = -2 \int_{-1}^{1} [a(\boldsymbol{\xi}) U_0^2]_{\xi_1 = 1} d\xi_2 + \iint_S (2a(\boldsymbol{\xi}) V_{1\xi_1}^{(1)} + a_{\xi_1}(\boldsymbol{\xi}) V_1^{(1)}) U_0 dS,$$
$$t_{12} = t_{21} = \frac{1}{2} \iint_S \left(2a(\boldsymbol{\xi})(V_{1\xi_2}^{(1)} + V_{1\xi_1}^{(2)}) + a_{\xi_2}(\boldsymbol{\xi}) V_1^{(1)} + a_{\xi_1}(\boldsymbol{\xi}) V_1^{(2)} \right) U_0 dS,$$
$$t_{22} = -2 \int_{-1}^{1} [a(\boldsymbol{\xi}) U_0^2]_{\xi_2 = 1} d\xi_1 + \iint_S (2a(\boldsymbol{\xi}) V_{1\xi_2}^{(2)} + a_{\xi_2}(\boldsymbol{\xi}) V_1^{(2)}) U_0 dS.$$

This is entirely on the macroscale with the microstructure built in through integrated quantities.

1D example: a piecewise string

If we take

$$l^2 \frac{d^2u}{dx^2} + \Omega^2 \frac{u}{c^2(\xi)} = 0, \qquad \text{with} \quad \Omega = \frac{\omega l}{\hat{c}_0}.$$

and apply the approach described above we get
$u \sim u_0(\xi, X) = f_0(X) U_0(\xi, \Omega_0)$, $\Omega^2 = \Omega_0^2 + \epsilon^2 \Omega_2^2 + ..$, where

$$T f_{0XX} + \Omega_2^2 f_0 = 0$$

and

$$T = 2\left(\frac{-U_0^2(1,\Omega_0) + A\int_{-1}^{1} U_0 V_{1\xi} d\xi}{\int_{-1}^{1} U_0^2/c^2(\xi) d\xi}\right), \qquad A = \frac{2U_0(1,\Omega_0)}{V_1(1,\Omega_0) - V_1(-1,\Omega_0)}$$

Note for Bloch waves Floquet-Bloch conditions lead to
$u(X+2\epsilon, \xi) = \exp(2i\epsilon\kappa) u(X, \xi)$ this forces $f_0(X) = \exp(i\kappa X)$. Thus the dispersion relation follows from

$$T\kappa^2 = \Omega_2^2$$

so locally quadratic. "Everything" is encapsulated in T.

If c piecewise so

$$c(\xi) = \begin{cases} 1/r \quad \text{for} \quad 0 \leq \xi < 1 \\ 1 \quad \text{for} \quad -1 \leq \xi < 0 \end{cases}$$

then can solve analytically

$$U_0(\xi, \Omega_\theta^{(n)}) = \begin{cases} \sin r\Omega_\theta^{(n)}\xi + p\cos r\Omega_\theta^{(n)}\xi \quad \text{for} \quad 0 \leq \xi < 1 \\ r\sin\Omega_\theta^{(n)}\xi + p\cos\Omega_\theta^{(n)}\xi \quad \text{for} \quad -1 \leq \xi < 0 \end{cases}$$

with $p = (r\sin\Omega_\theta^{(n)} \pm \sin r\Omega_\theta^{(n)})/(\cos\Omega_\theta^{(n)} \mp \cos r\Omega_\theta^{(n)})$.

$$V_1(\xi, \Omega_\theta^{(n)}) = \begin{cases} \sin r\Omega_\theta^{(n)}\xi \quad \text{for} \quad 0 \leq \xi < 1 \\ r\sin\Omega_\theta^{(n)}\xi \quad \text{for} \quad -1 \leq \xi < 0. \end{cases}$$

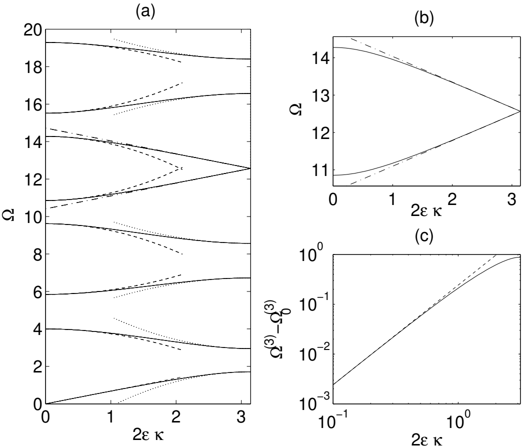
To get

$$T_\theta^{(n)} = \pm 4\Omega_\theta^{(n)} \frac{\sin\Omega_\theta^{(n)}\sin\Omega_\theta^{(n)}r}{(r\sin\Omega_\theta^{(n)} \mp \sin r\Omega_\theta^{(n)})(\cos\Omega_\theta^{(n)} \mp \cos r\Omega_\theta^{(n)})}$$

Dispersion curves

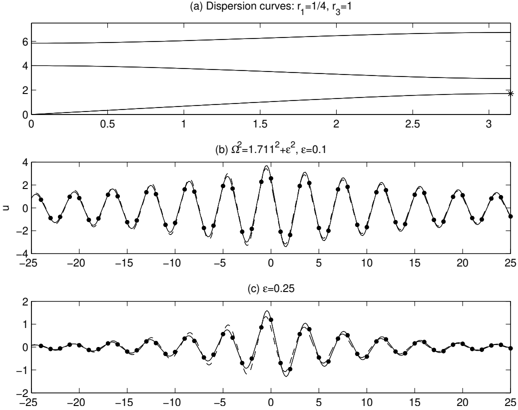
Verify versus the Bloch dispersion relation which is known exactly in this example as

$$2r[\cos\Omega\cos r\Omega - \cos 2\epsilon\kappa] - (1+r^2)\sin\Omega\sin r\Omega = 0.$$



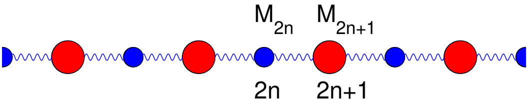
Line forcing

We can naturally start using the continuum model to solve line forcing (or defect or other forcing etc) on the long-scale and doing so is much easier than solving the full problem. In this example for a delta function forcing at $x = -1$



Discrete media: localization of modes

Return to masses and springs:



The displacements y_{2n} satisfy

$$y_{2n-1} + y_{2n+1} - 2y_{2n} = -M_{2n}\Omega^2 y_{2n}$$

$$y_{2n} + y_{2n+2} - 2y_{2n+1} = -M_{2n+1}\Omega^2 y_{2n+1}.$$

Now let the masses vary on a long-scale (defined later..)

$$M_{2n} = M_2[1 + \gamma g_{2n}], \qquad M_{2n+1} = M_1[1 + \gamma g_{2n+1}]$$

Do localised/ trapped modes exist? Does the theory find them?

Two scales

A discrete version of the theory: long scale $N \gg 1$ and $\epsilon = 1/N \ll 1$. The long variable $\eta = 2n/N$ is continuous. Take an elementary cell of four masses $2n$, $2n+1$ and their neighbours, call these $m = -1, 0, 1, 2$.

$$y_{2n+m} = y(\eta + m\epsilon, m) \sim y(\eta, m) + m\epsilon y_\eta(\eta, m) + \frac{(m\epsilon)^2}{2} y_{\eta\eta}(\eta, m) + \dots$$

The four cell masses are at $y_{2n-1} = y(\eta - \epsilon, -1)$, $y_{2n} = y(\eta, 0)$, $y_{2n+1} = y(\eta + \epsilon, 1)$ and $y_{2n+2} = y(\eta + 2\epsilon, 2)$.
In fact just need two of them and use that they are in-phase or out-of-phase over a cell

$$[y_{2n-1}, y_{2n+2}] = [y(\eta - \epsilon, -1), y(\eta + 2\epsilon, 2)] = (-1)^J [y(\eta - \epsilon, 1), y(\eta + 2\epsilon, 0)]$$

Take

$$M_{2n} = M_2[1 + \epsilon^2 \alpha g(\eta)], \qquad M_{2n+1} = M_1[1 + \epsilon^2 \alpha g(\eta)].$$

This all reduces to a matrix problem

$$[A_0 - \Omega^2 M(1 + \epsilon^2 \alpha g(\eta)) + \epsilon A_1(\partial, \Omega) + \epsilon^2 A_2(\partial^2, \Omega)]\mathbf{y}(\eta) = 0,$$

where ∂ denotes $\partial/\partial\eta$, $\mathbf{y}(\eta) = [y(\eta, 0),\ y(\eta, 1)]^T$. When A_0, A_1 and A_2 are matrix differential operators.
The ansatz

$$\mathbf{y}(\eta) = \mathbf{y_0}(\eta) + \epsilon \mathbf{y_1}(\eta) + \epsilon^2 \mathbf{y_2}(\eta) + \dots \tag{0.13}$$

$$\Omega^2 = \Omega_0^2 + \epsilon \Omega_1^2 + \epsilon^2 \Omega_2^2 + \dots \tag{0.14}$$

and again solve order-by-order. Skip the algebra...
Note simply that

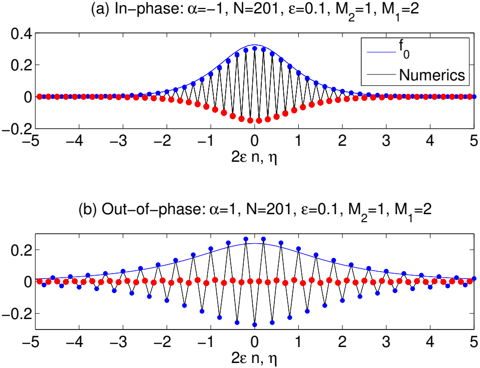
$$\mathbf{y}_0(\eta) = f_0(\eta)\mathbf{Y}_0$$

and again a differential ODE for f_0 appears...

ODEs for optical mode

In-phase/ out-of phase (if $\alpha = 0$ get Bloch asymptotics).

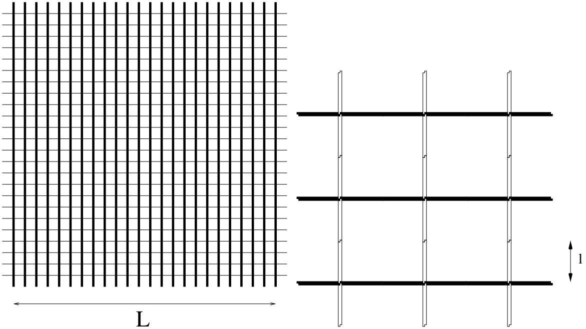
$$\frac{2}{(M_1 \pm M_2)} f_{0\eta\eta} \mp [\Omega_2^2 + \alpha\Omega_0^2 g(\eta)] f_0 = 0,$$



Localised modes for $M_1 = 2$, $M_2 = 1$ showing the numerical solutions versus the f_0 from the asymptotic equations with $\operatorname{sech}^2(\eta)$ variation.

Frames

Models in structural dynamics: frames, trusses, nets - cellular solids (bones, foams ...).



A macrostructure net of overlapping strings on scale L (left) constructed from an elementary cell of microscale l (right).

Homogenized equation

Follow the multiple scales procedure to get

$$T_{11} f_{0XX} + T_{22} f_{0YY} + \Omega_2^2 f_0 = 0 \tag{0.15}$$

where

$$T_{11} = \frac{4 r_1 r_3 \Omega_0}{Q_1 I}, \qquad T_{22} = \frac{4 r_2 r_4 \Omega_0}{Q_2 I} \tag{0.16}$$

where I is the integral

$$I = r_1^2 \int_0^1 \left(U_0^{(1)}\right)^2 d\xi + r_3^2 \int_{-1}^0 \left(U_0^{(3)}\right)^2 d\xi + r_2^2 \int_0^1 \left(U_0^{(2)}\right)^2 d\eta + r_4^2 \int_{-1}^0 \left(U_0^{(4)}\right)^2 d\eta \,.$$

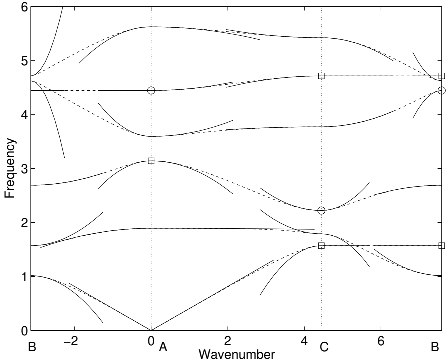
and

$$Q_1 = r_1 \cos \Omega r_1 \sin \Omega r_3 + r_3 \cos \Omega r_3 \sin \Omega r_1$$

(Q_2 similar).

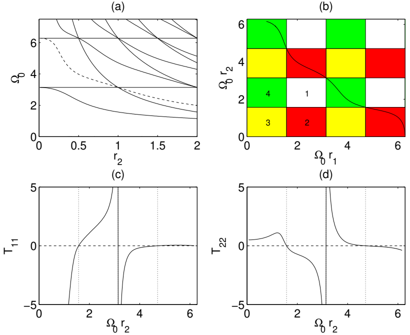
The sign of T_{11}, T_{22} changes and we go from elliptic to hyperbolic equations - some directions are "weak". Can use this to generate asymptotic Bloch curves, localization, forcing etc.

Typical dispersion relation



The first six dispersion curves $(r_1, r_2, r_3, r_4) = (1, 1/\sqrt{2}, 1, 1/\sqrt{2})$ with the full numerics as dashed curves and the solid lines from the asymptotics. The squares (circles) denote degenerate cases where $Q_1 = 0$ ($Q_2 = 0$); those along C and B arise at $\Omega_0 r_1 = \pi/2, 3\pi/2$ etc.

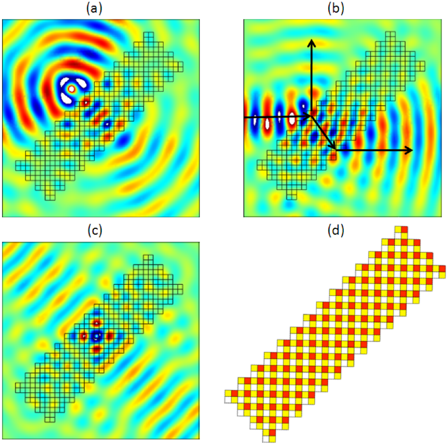
Variation of T



Variation of T_{11}, T_{22} with parameters for periodic standing waves and $r_1 = r_3 = 1$ and $r_2 = r_4$. (a) standing wave eigenvalue, Ω_0, with r_2; (b) $\Omega_0 r_1$ versus $\Omega_0 r_2$. (b) colour-coded regions with 1 having both T_{11}, T_{22} negative, 2 has T_{11} negative and T_{22} positive, 3 has both T_{11}, T_{22} positive and 4 has T_{11} positive and T_{22} negative. (c) and (d) T_{11} and T_{22} versus $\Omega_0 r_2$ for solutions in (b). Dashed line is zero and dotted show $\Omega_0 r_2 = \pi/2, \pi, 3\pi/2$ with sign changes in T_{11}, T_{22} occuring in (b).

Photonics- checkerboard media

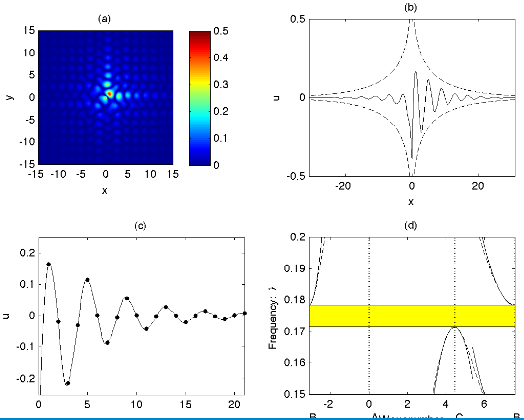
An idealised medium made of checkerboards, each with a different refractive index. This illustrates interesting features such as all-angle-negative refraction and ultra-refraction. These only occur for very precise frequencies that can now be predicted by the theory.



Localised defect mode

Excitation in a stop-band, note periodic-periodic behaviour and envelope given by asymptotic theory

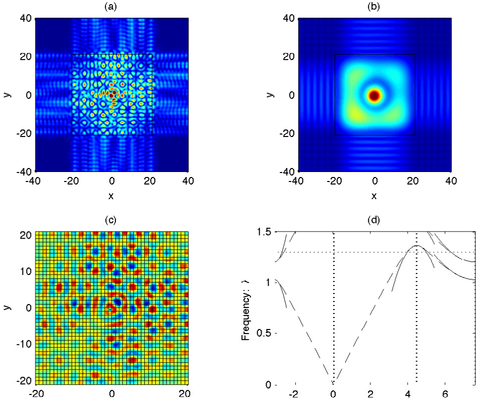
$$u(x,y) \sim \frac{1}{2\pi} K_0 \left(\sqrt{\frac{\Omega^2 - \Omega_0^2}{|T|}} \sqrt{x^2 + y^2} \right)$$



Ultra-refraction

Excitation close to a standing wave frequency where locally the group velocity is almost zero - the medium is then "slow" and omni-directional antennae can be designed. Theory gives effective refractive index

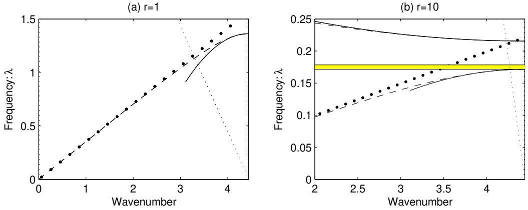
$$n_{eff}(\Omega, \Omega_0) = \frac{2}{\sqrt{|T|}} \frac{(\Omega^2 - \Omega_0^2)}{\Omega},$$



All-angle-negative refraction

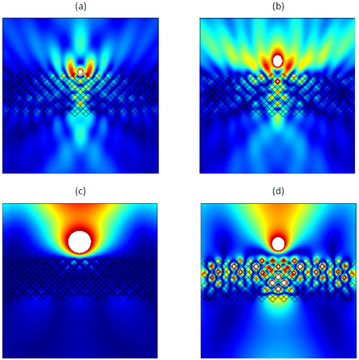
More complicated, here one needs a crossing of the dispersion curves with a stright line and the theory predicts the critical frequency is

$$\Omega_{HFH} = \frac{\Omega_0}{\sqrt{1 - T/4}}.$$



Concentrate on $r = 10$ case (high contrast)

All-angle-negative refraction - imaging



Top panels $r = 1$ (low contrast), upper panels $r = 10$ case (high contrast). Note the slab behaves as if it has negative refractive index.

Some references

The homogenization is in Craster, Kaplunov and Pichugin (Proc R Soc Lond A, 2010).
For discrete media in Craster, Kaplunov & Postnova (QJMAM, 2010).
For nets in Nolde, Craster, Kaplunov (JMPS 2011) and
For optics in Craster, Kaplunov, Nolde and Guenneau (JOSA 2011 & Wave Motion 2012).
Elastic plates in Antonakakis & Craster (Proc R Soc Lond 2012).