Dispersion of Flexural Waves

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Outline of the Lecture

I. Flexural Waves in Plates

1. Low-Frequency Limit of Lamb Waves
2. Flexural Wave Equation

II. Dispersion of waves

1. Wave packet, Phase Velocity, Group Velocity
2. Propagation in Dispersive Media
3. Dispersion in Plates
   a) Isotropic Plate
   b) Orthotropic Plate

Low-Frequency limit of Lamb Waves

Dispersion relation:

$$\frac{\omega^4}{c_T^4} = 4k^2q^2\left(1 - \frac{p}{q}\frac{\tan(ph/2+\alpha)}{\tan(qh/2+\alpha)}\right) \quad \text{with} \quad \alpha = 0 \quad \text{or} \quad \pi/2$$

At low frequency, first order development: $\omega \to 0, k \to 0, p \to 0, q \to 0$

For $\alpha = 0,$

$$\frac{\omega^4}{c_T^4} = 4k^2(q^2 - p^2) = 4k^2(k_T^2 - k_L^2) = 4k^2\omega^2\left(\frac{1}{c_T^2} - \frac{1}{c_L^2}\right)$$

$$c_P = \frac{\omega}{k} = 2c_T\sqrt{1 - \frac{c_T^2}{c_L^2}} = c_T\sqrt{\frac{2}{1-\nu}} = c_L\frac{\sqrt{1-\nu}}{1-\nu} \quad \Rightarrow \quad \sqrt{2}c_T < c_P < c_L$$

Symmetric Mode

Displacements:

$$\begin{cases} u_x = \left(i k B \cos(pz) - q A \cos(qz) \right) e^{i(kx - \omega t)} \\ u_z = \left(p B \sin(pz) + i k A \sin(qz) \right) e^{i(kx - \omega t)} \end{cases}$$

At low frequency,

$$\begin{cases} p^2 = \frac{\omega^2}{c_L^2} - k^2 = k^2 \left(\frac{c_P^2}{c_L^2} - 1 \right) \\ q^2 = \frac{\omega^2}{c_T^2} - k^2 = k^2 \left(\frac{c_P^2}{c_T^2} - 1 \right) \end{cases}$$

$$\sqrt{2} c_T < c_P < c_L, \quad \begin{cases} p = i k \sqrt{1 - \frac{c_P^2}{c_L^2}} \\ q = k \sqrt{\frac{c_P^2}{c_T^2} - 1} \end{cases}$$

Symmetric Mode

Displacements:

$$\begin{cases} u_x = (ikB - qA)\, e^{i(kx-\omega t)} \\ u_z = (Bp^2 + iAkq)\, z\, e^{i(kx-\omega t)} \end{cases}$$

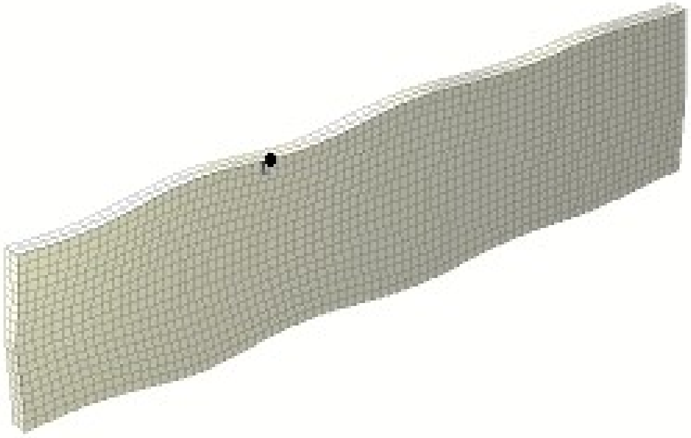
$$B = -2ik \frac{q}{k^2 - q^2} A$$

$$\begin{cases} p = ik\sqrt{1 - \dfrac{c_P^2}{c_L^2}} \\ q = k\sqrt{\dfrac{c_P^2}{c_T^2} - 1} \end{cases}$$

After simplifications:

$$\begin{cases} u_x = q\dfrac{A}{\nu} e^{i(kx-\omega t)} \\ u_z = -i\dfrac{qkAz}{1-\nu} e^{i(kx-\omega t)} \end{cases}$$

$$\frac{|u_z(\pm h/2)|}{u_x} = \frac{1-\nu}{\nu}\frac{kh}{2} \ll 1$$



Antisymmetric Mode

Displacements:

$$\begin{cases} u_x = (ikBpz - qAqz)\, e^{i(kx-\omega t)} \\ u_z = (pB + ikA)\, e^{i(kx-\omega t)} \end{cases}$$

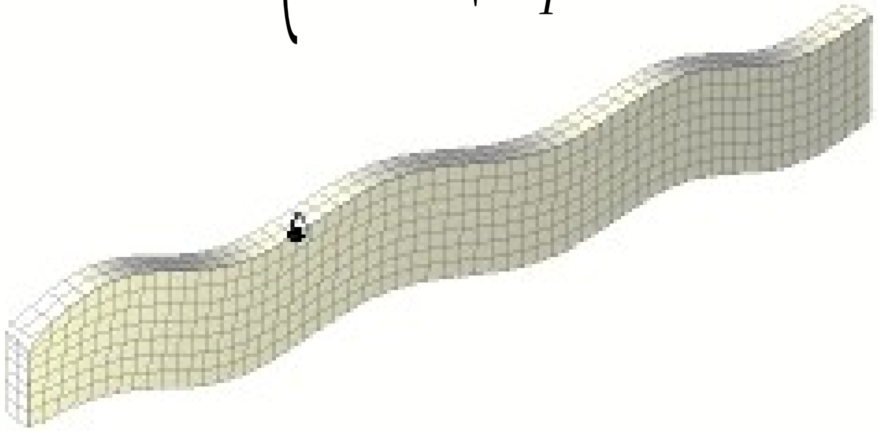
$$B = -2ik \frac{q}{k^2-q^2} A$$

$$\begin{cases} p = ik\sqrt{1-\dfrac{c_P^2}{c_L^2}} \\ q = k\sqrt{\dfrac{c_P^2}{c_T^2}-1} \end{cases}$$

After simplifications:

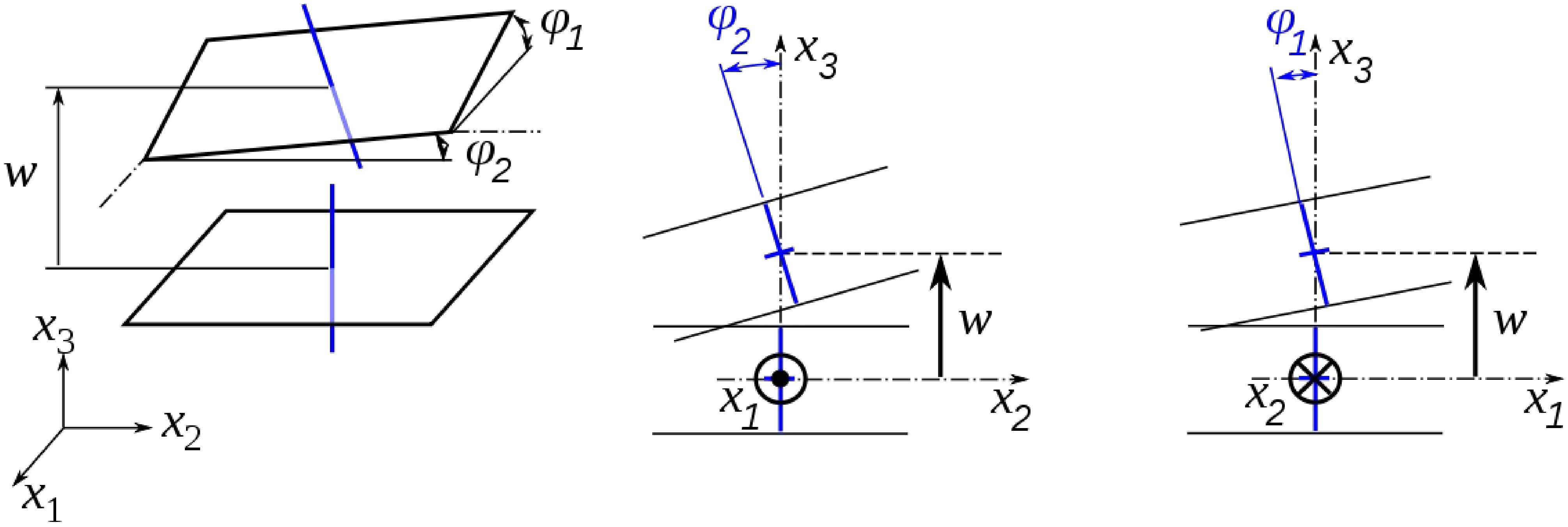
$$\begin{cases} u_x = -k^2 A \dfrac{k^2+q^2}{k^2-q^2}\, z\, e^{i(kx-\omega t)} \\ u_z = ikA \dfrac{k^2+q^2}{k^2-q^2}\, e^{i(kx-\omega t)} \end{cases}$$

$$\frac{|u_x(\pm h/2)|}{u_z} = \frac{kh}{2} \ll 1$$



Kirchoff-Love Plate

- Hypothesis of pure flexion from start
- Write the corresponding stress and deformation tensors
- Derivation from Newton's laws or Hamiltonian principle



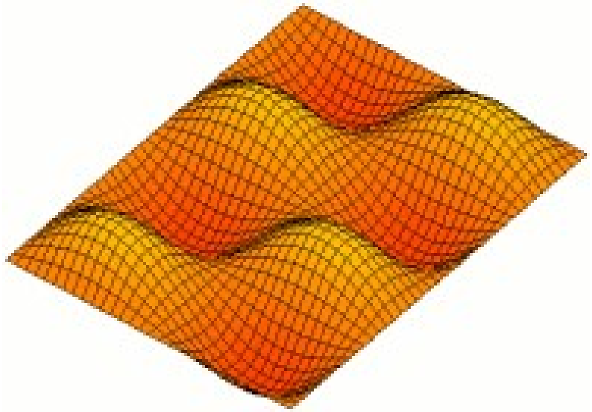
Kirchhoff-Love Plate

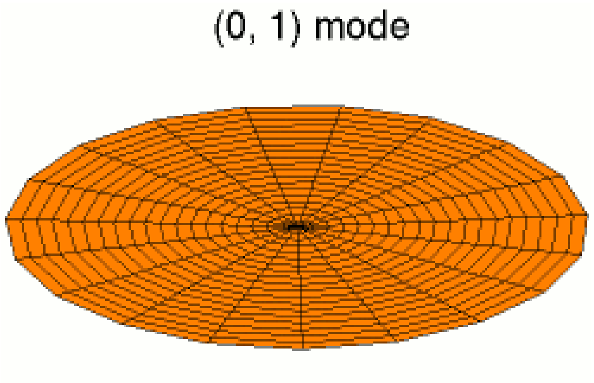
- The transverse displacement is dominant, so we only need: $u_z = u$

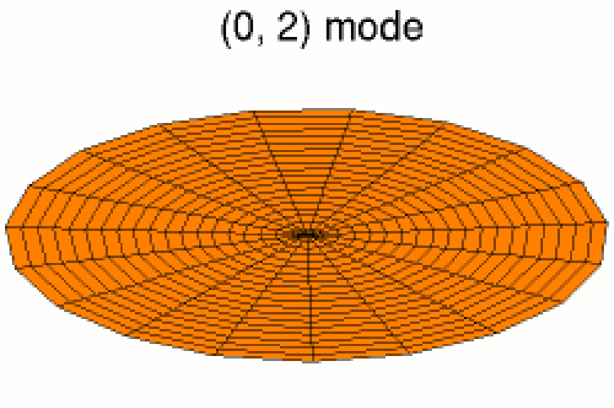
$$\rho h \frac{\partial^2 u}{\partial t^2} + D \Delta^2 u = 0,$$

$$\text{with} \quad \Delta^2 = \nabla^4 = \frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4} + 2 \frac{\partial^4}{\partial x^2 \partial y^2} \quad \text{and} \quad D = \frac{E h^3}{12\left(1 - \nu^2\right)}$$

- Order 4 operator, more boundary conditions needed.
- Analytical solutions for specific geometries in finite structures







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Dispersion of Waves

- Relation between frequency and wavenumber:

$$f(\omega, k) = 0, \quad \text{or} \quad \omega = f(k)$$

Exemple: D'Alembert equation: $\dfrac{\partial^2 u}{\partial x^2} - \dfrac{1}{c^2}\dfrac{\partial^2 u}{\partial t^2} = 0,$

General Solution: $u(x,t) = u_1(x - ct) + u_2(x + ct)$

- Generic method to find the dispersion : Progressive Harmonic Wave Decomposition $e^{i(kx - \omega t)}$

Dispersion Relation

Injecting the progressive harmonic form in the wave equation:

$$\frac{\partial}{\partial x} = ik$$
$$\frac{\partial}{\partial t} = -i\omega$$

$$k^2 e^{i(kx-\omega t)} - \frac{\omega^2}{c^2} e^{i(kx-\omega t)} = 0$$

$$\boxed{k^2 = \frac{\omega^2}{c^2}}$$

Linear relation between k and ω

Dispersion Relation

Flexion in a beam:

$$\rho S \frac{\partial^2 w}{\partial t^2} + E I \frac{\partial^4 w}{\partial x^4} = 0$$

$$-\rho S \omega^2 e^{i(kx - \omega t)} + E I k^4 e^{i(kx - \omega t)} = 0$$

$$\omega^2 = \frac{E I}{\rho S} k^4$$

Non linear relation between k and ω

Phase Velocity

The phase of the progressive harmonic wave is:

$$\Psi = k\,x - \omega\,t$$

We look for the positions where $\Psi = \mathrm{cte}$ for a given t:

$$x = \frac{\omega}{k}t - \frac{\Psi}{k}$$

This defines a plane in space, travelling at speed: $v_\phi = \dfrac{\omega}{k}$

Represents the speed of a given harmonic **plane wave**.

Phase Velocity

Medium is non dispersive if $v_\phi = \text{cte}$

Example: for acoustic waves, $v_\phi(\omega) = c = \dfrac{1}{\sqrt{\rho\,\chi}}$

Also true for all waves following d'Alembert equation

All frequencies travel at the same speed v_ϕ

Wave Packet

All the harmonic plane waves $e^{i(kx-\omega t)}$ are solutions of the wave equation if they verify the dispersion relation.

By linearity, plane waves can be combined:

$$u(x,t)=\sum_n A(\omega_n)e^{i(k_n x-\omega_n t)}$$

$$u(x,t)=\int a(\omega)e^{i(kx-\omega t)}d\omega$$

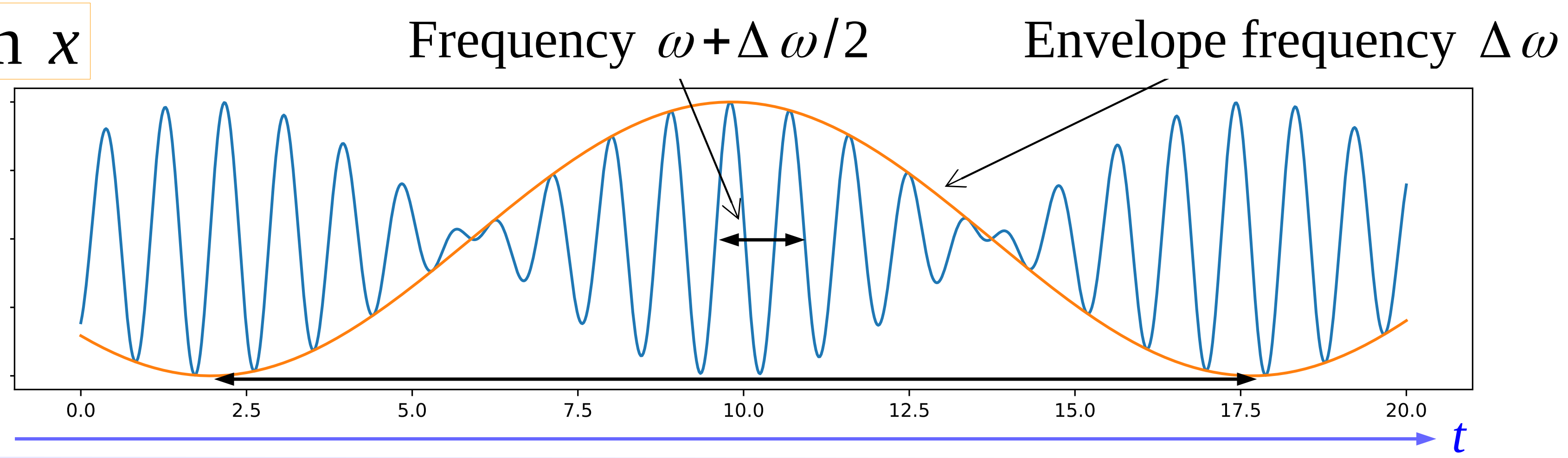
Wave Packet

Wave packet with two harmonic waves:

$$u(x,t)=\cos(kx-\omega t)+\cos((k+\Delta k)x-(\omega+\Delta\omega)t)$$

$$u(x,t)=2\cos\left((k+\Delta\frac{k}{2})x-(\omega+\Delta\frac{\omega}{2})t\right)\cos\left(\frac{\Delta k}{2}x-\frac{\Delta\omega}{2}t\right)$$

fixed position x



Group Velocity

$$u(x,t)=2\cos\left(\left(k+\Delta\frac{k}{2}\right)x-\left(\omega+\Delta\frac{\omega}{2}\right)t\right)\cos\left(\frac{\Delta k}{2}x-\frac{\Delta\omega}{2}t\right)$$

The fast oscillation travels at a speed:

$$v_\phi=\frac{\omega+\frac{\Delta\omega}{2}}{k+\frac{\Delta k}{2}}$$

The envelope travels at speed:

$$v_g=\frac{\Delta\omega}{\Delta k}$$

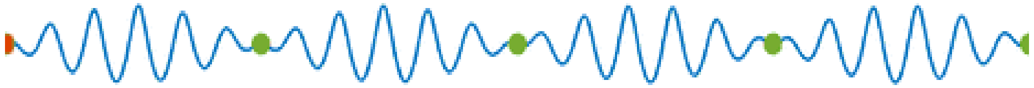
Group Velocity

$$u(x,t)=2\cos\left(\left(k+\Delta\frac{k}{2}\right)x-\left(\omega+\Delta\frac{\omega}{2}\right)t\right)\cos\left(\frac{\Delta k}{2}x-\frac{\Delta\omega}{2}t\right)$$

The fast oscillation travels at a speed:

$$v_\phi=\frac{\omega+\frac{\Delta\omega}{2}}{k+\frac{\Delta k}{2}}\rightarrow\frac{\omega}{k}$$

The envelope travels at speed:

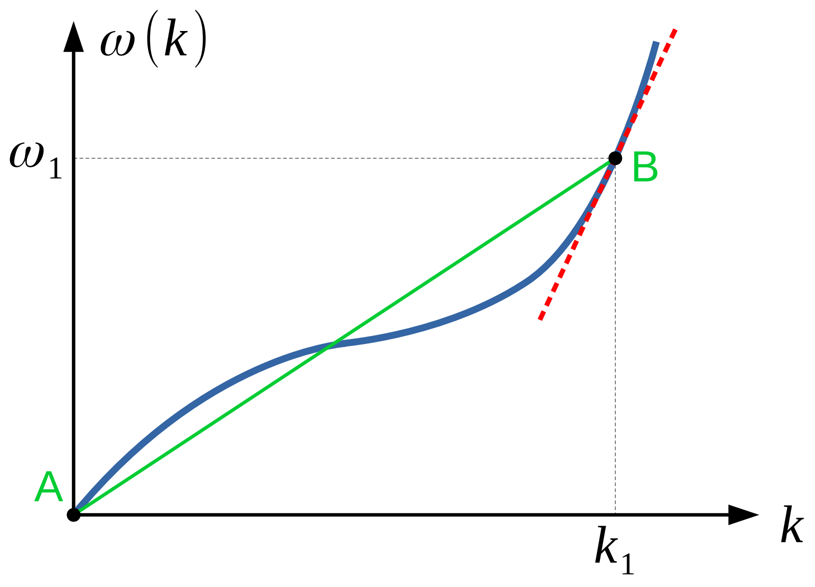


Group Velocity Animation

$$v_g=\frac{\Delta\omega}{\Delta k}\rightarrow\frac{d\omega}{dk}$$

Group Velocity

Example of dispersion curve:



v_ϕ : slope of the string AB

v_g : slope of the tangent

v_g can be negative or zero !

Zero Group Velocity Modes in Lamb waves

Beam Flexion Dispersion

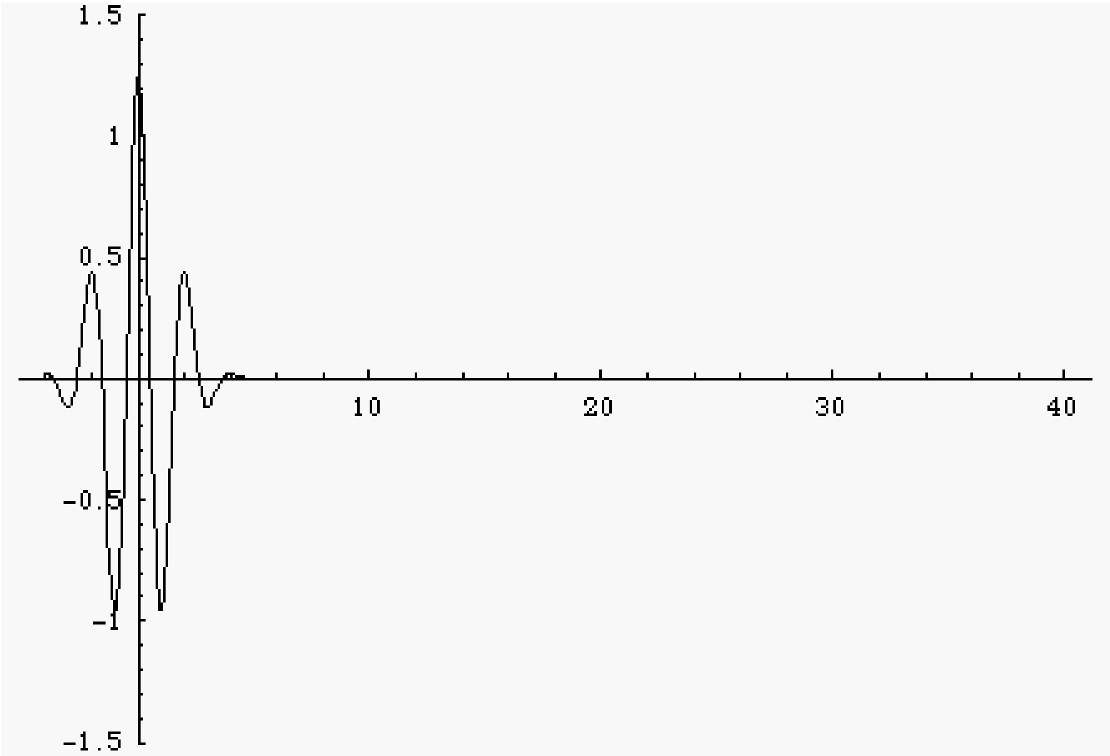
Dispersion curve of beam flexural waves:

$$\omega = k^2 \sqrt{\frac{E I}{\rho S}}$$

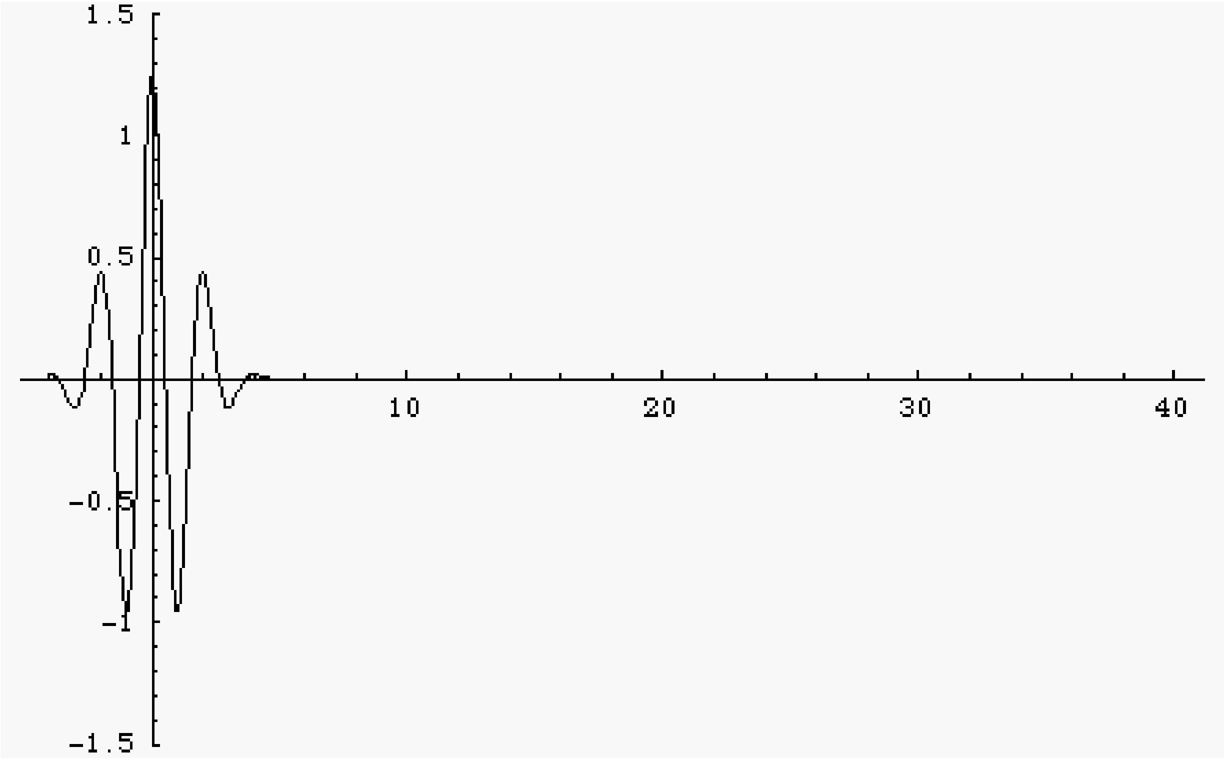
$$v_\phi = \frac{\omega}{k} = k \sqrt{\frac{E I}{\rho S}} = \left(\frac{E I}{\rho S} \right)^{1/4} \sqrt{\omega} \qquad v_g = \frac{d\omega}{dk} = 2 k \sqrt{\frac{E I}{\rho S}} = 2 v_\phi$$

- Higher frequencies travel faster
- Different frequencies are **dispersing**
- A wave packet is spreading through propagation

Dispersion of a Wave Packet

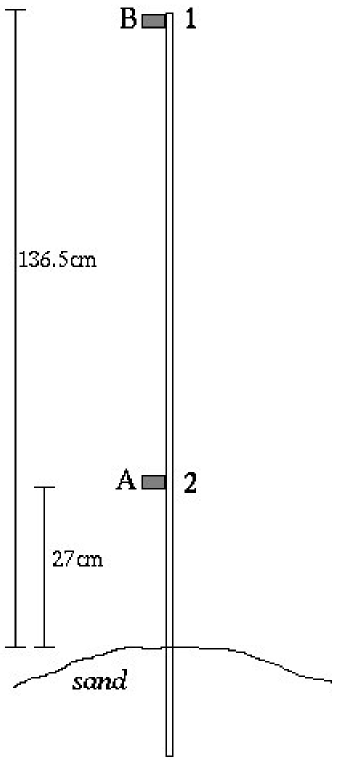


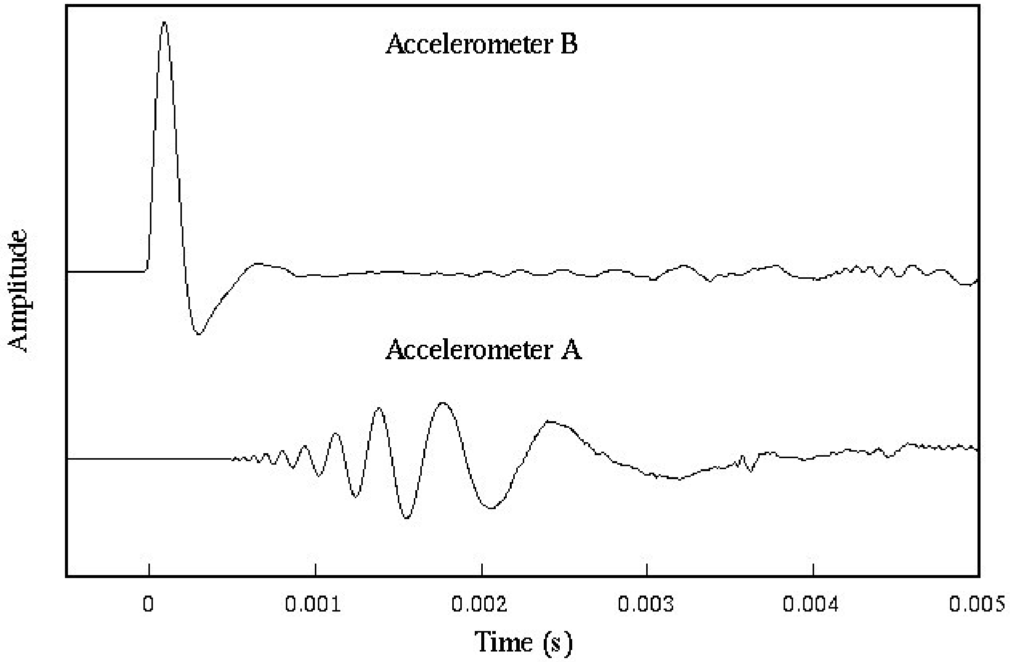
Non-dispersive medium



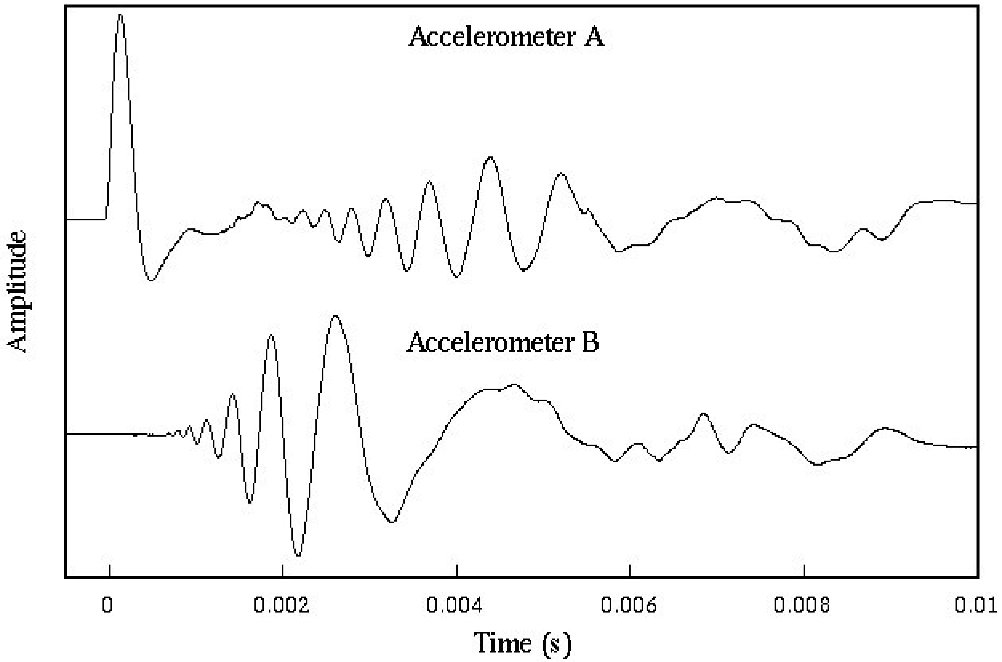
Dispersive medium

Group Velocity





Hammer impact at Point B



Hammer impact at Point A

Dispersion in Plates: Isotropic Case

Flexural waves equation:

$$\rho h \frac{\partial^2 u}{\partial t^2} + D\left(\frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2}\right) = 0$$

Plane wave in 2D:

$$e^{i(\vec{k}\vec{r} - \omega t)} = e^{i(k_x x + k_y y - \omega t)} \qquad \vec{k} = k_x \vec{e}_x + k_y \vec{e}_y$$

$$\rho h \omega^2 - D\left(k_x^4 + k_y^4 + 2k_x^2 k_y^2\right) = 0$$

$$\omega^2 = \frac{D}{\rho h}\left(k_x^2 + k_y^2\right)^2$$

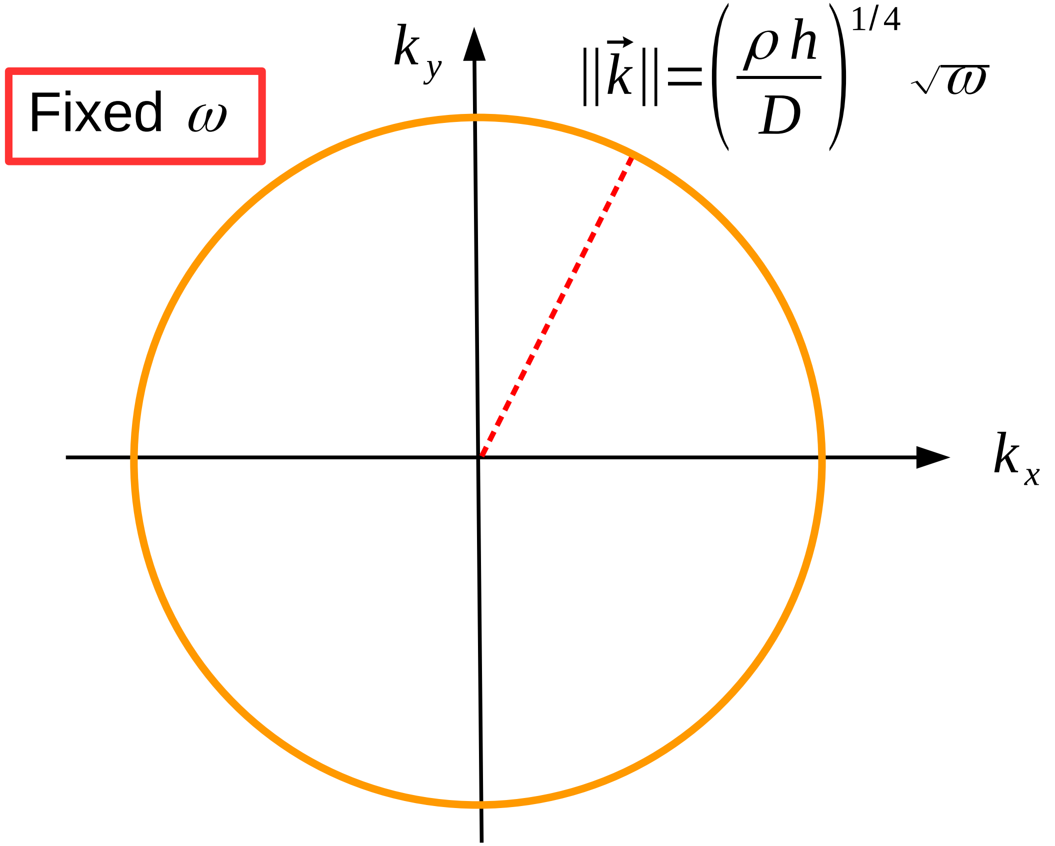
3 parameters

Dispersion in Plates: Isotropic Case

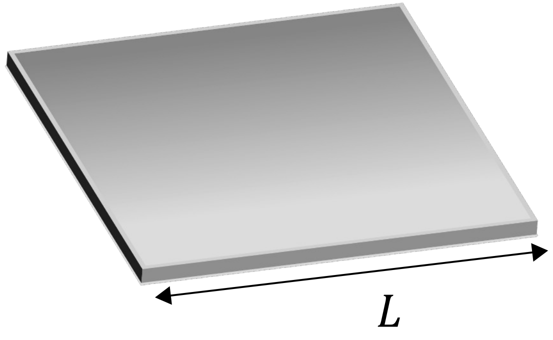
$$\omega=\sqrt{\frac{D}{\rho h}}\left(k_x^2+k_y^2\right)=\sqrt{\frac{D}{\rho h}}\|\vec{k}\|^2$$

Circle in the plane $\omega=$ cte

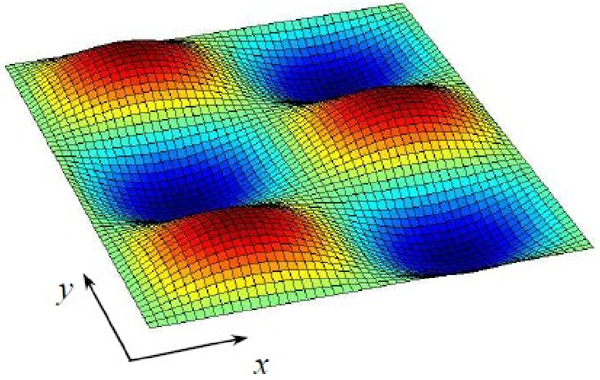
In 3D, paraboloid $\omega=f\left(k_x,k_y\right)$

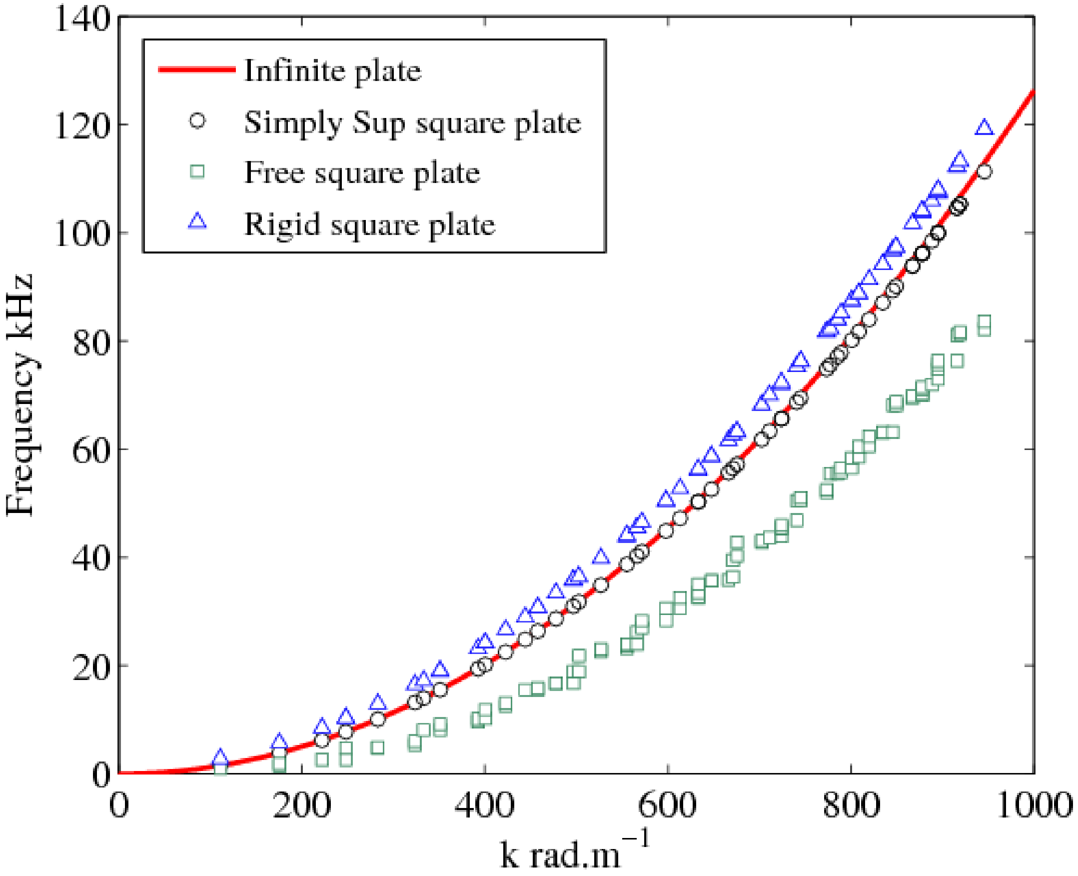


Finite Size Effects



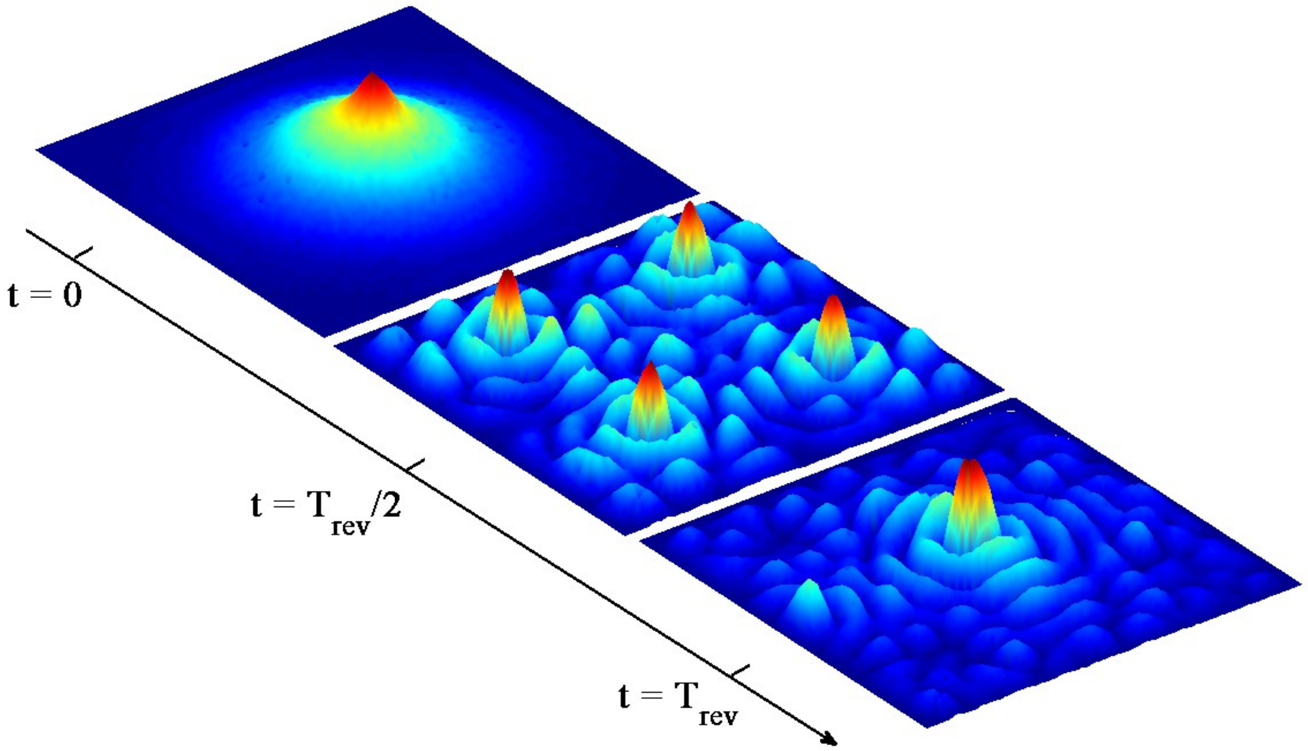
Duraluminium plate
dimensions : 40x40x0.5mm





- Discrete spectrum
- Deviation from the infinite medium

Revival Effect in Plates

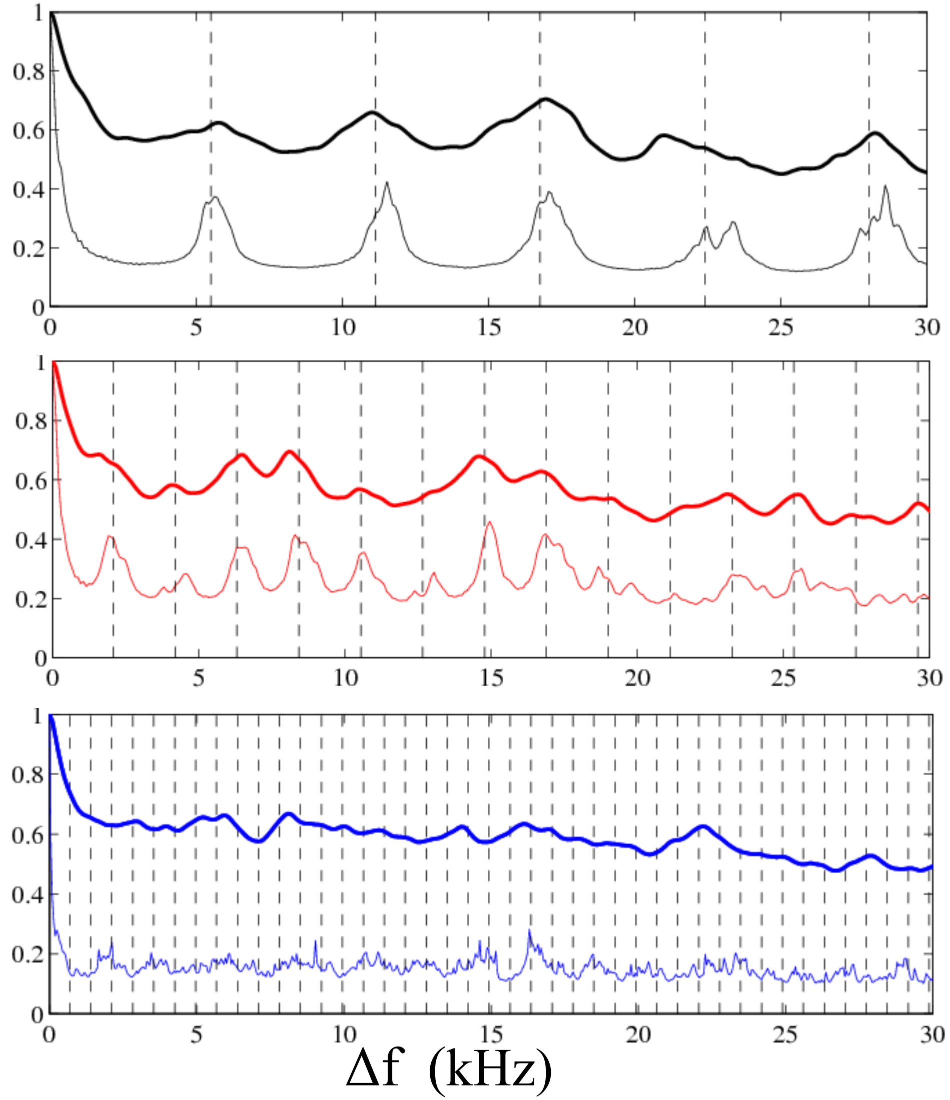


$$T_{rev} = \frac{4L^2}{\pi \frac{\partial^2 \omega_n}{\partial k^2}} = \frac{4\pi^2}{L^2}\sqrt{\frac{D}{\rho h}}$$

Revival Experiment Video

Revival Effect in Plates

Spectrum
Autocorrelation



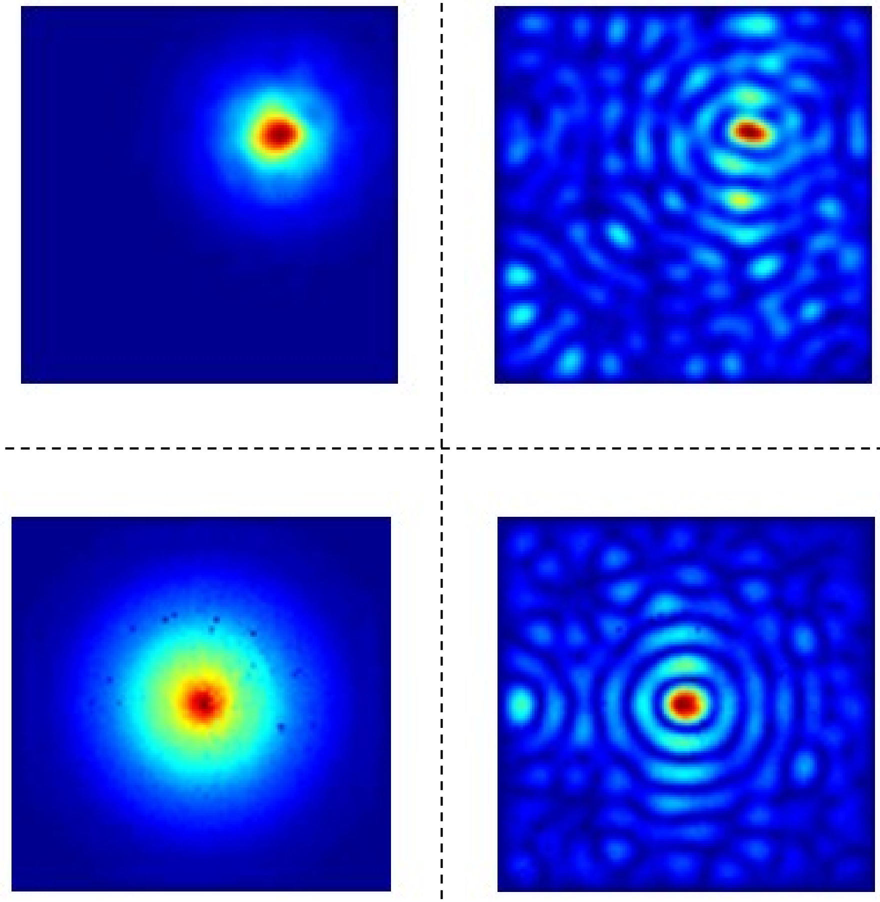
Source at centre
gcd(Δf)= 5.49kHz
T_{rev} = 182µs

1/3 diagonal
gcd(Δf)= 2.06kHz
T_{rev} = 484µs

Arbitrary position
gcd(Δf)= 0,69kHz
T_{rev} = 1450µs

Revival Effect in Plates

1st position



Source at
the centre

$$T_{rev} = 490\,\mu s$$

$$T_{rev} = 185\,\mu s$$

Theoretically,

$$T_{rev} = \frac{4\pi^2}{L^2}\sqrt{\frac{D}{\rho h}} = 1450\,\mu s$$

Quantum revival for elastic waves in thin plate
Dubois M. et al., Eur. Phys. Journ. S.T. 2017

Gradient-Index Lens

Maxwell fish-eye lens

Maxwell, Cambridge Dublin Math. J. (1854)

Refractive index variation:

$$n(r)=\frac{2}{\left(1+(r/r_0)^2\right)^2}$$

Kirchhoff-Love equation:

$$\rho h \frac{\partial^2 u}{\partial t^2}+D\Delta^2 u=0$$

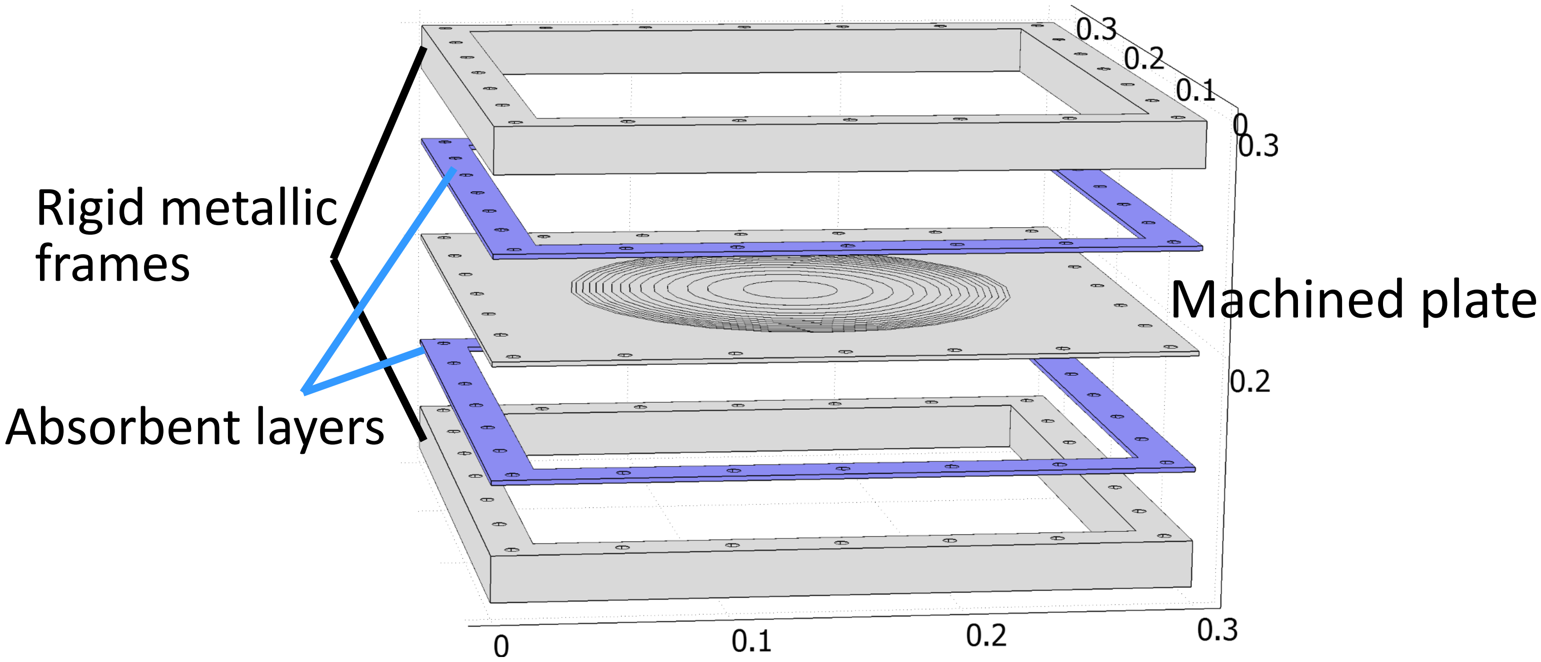
Phase velocity:

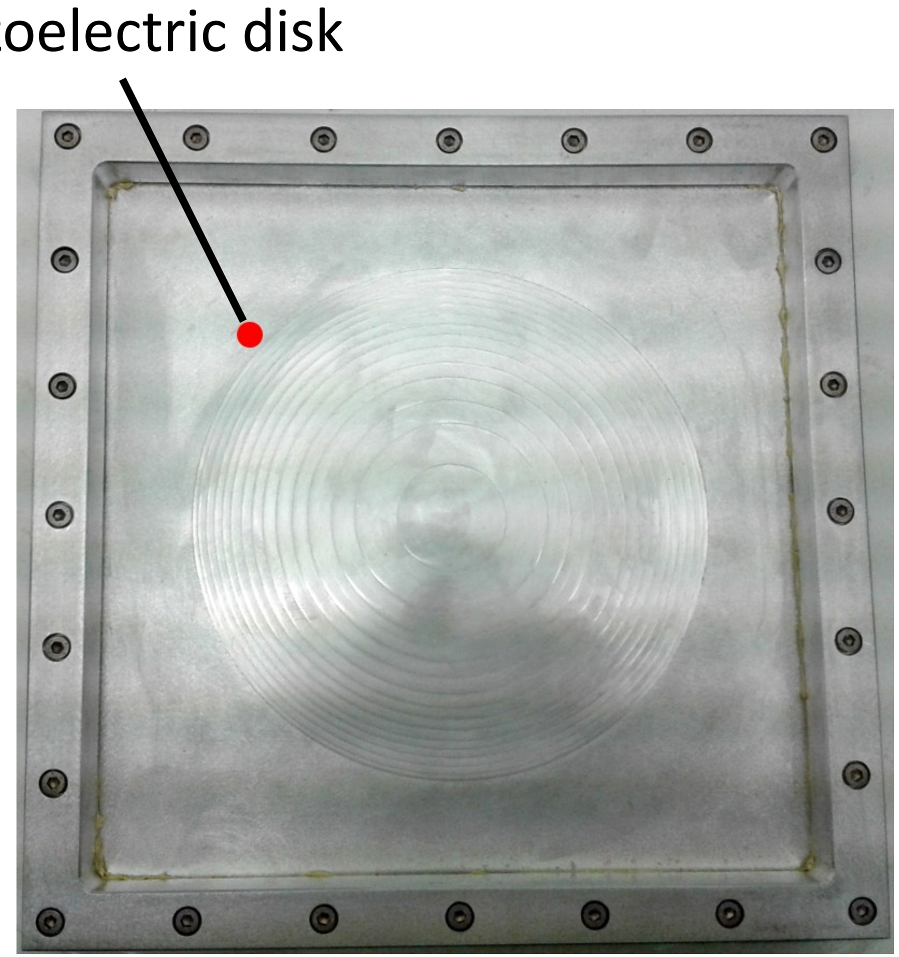
$$v_\phi=\left(\frac{E h^2 \omega^2}{12\rho(1-\nu^2)}\right)^{1/4}$$

Thickness profile:

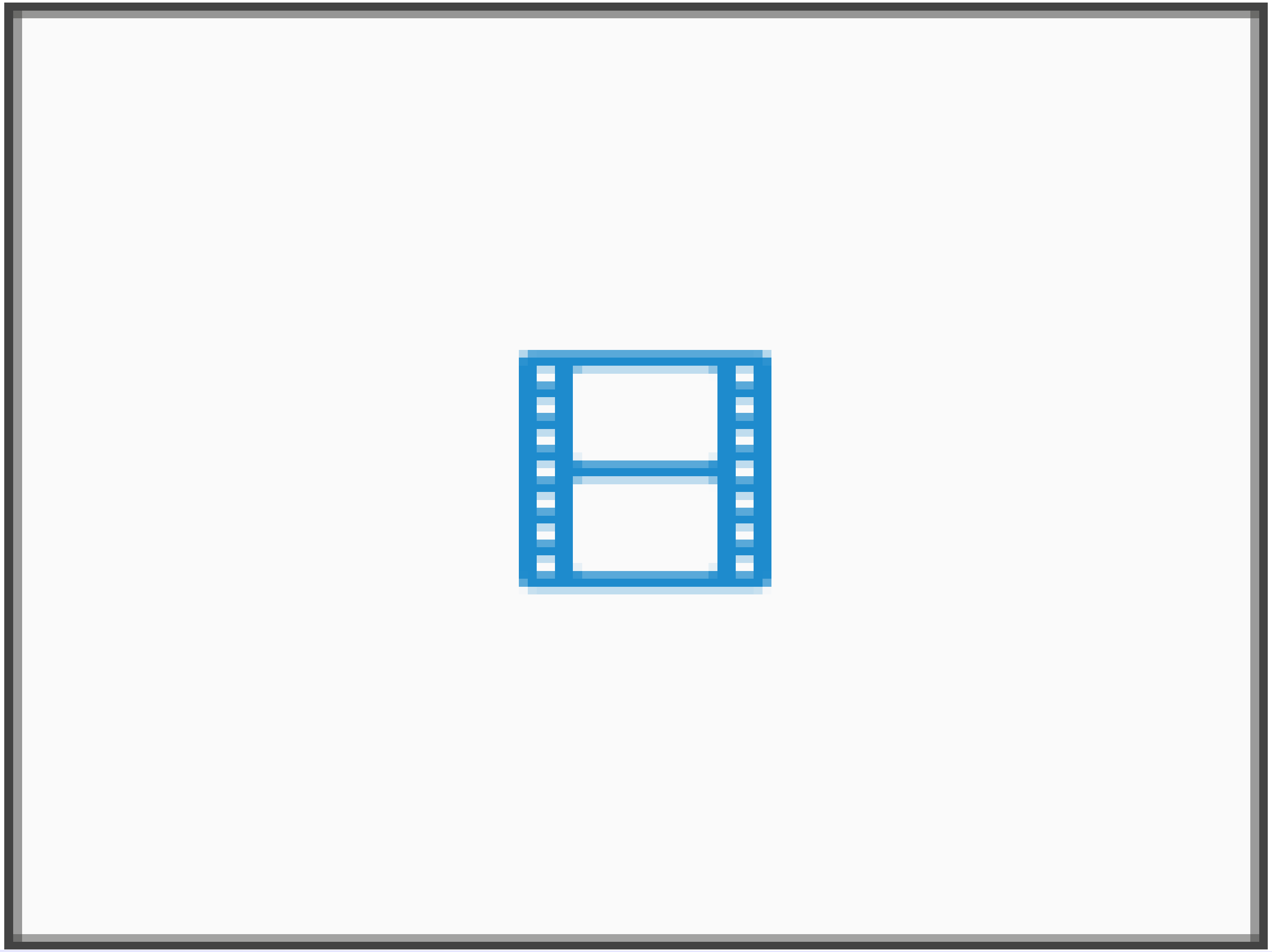
$$h(r)=h_0\left(1+(r/r_0)^2\right)^2$$

Gradient-Index Lens



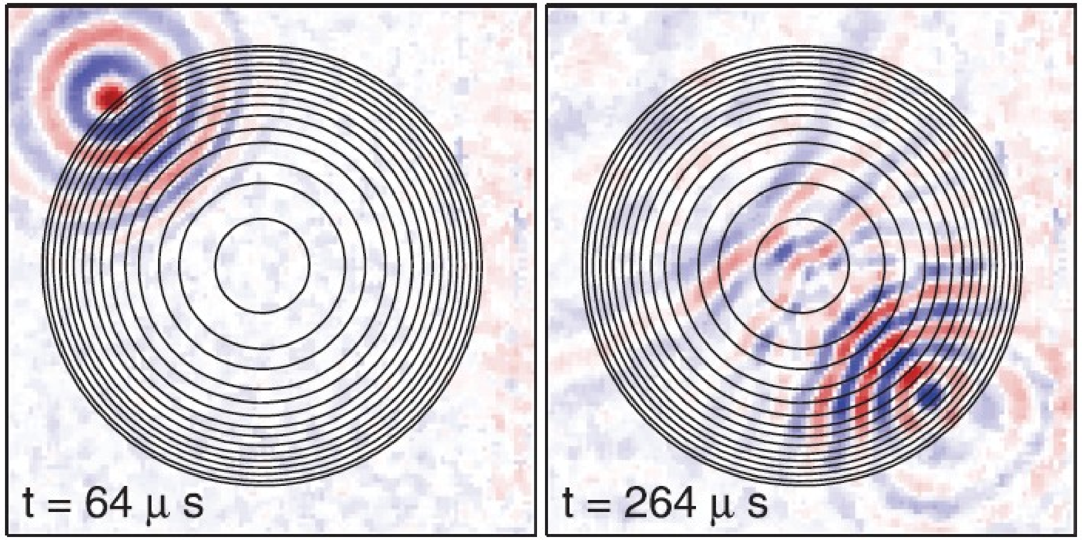


Gradient-Index Lens

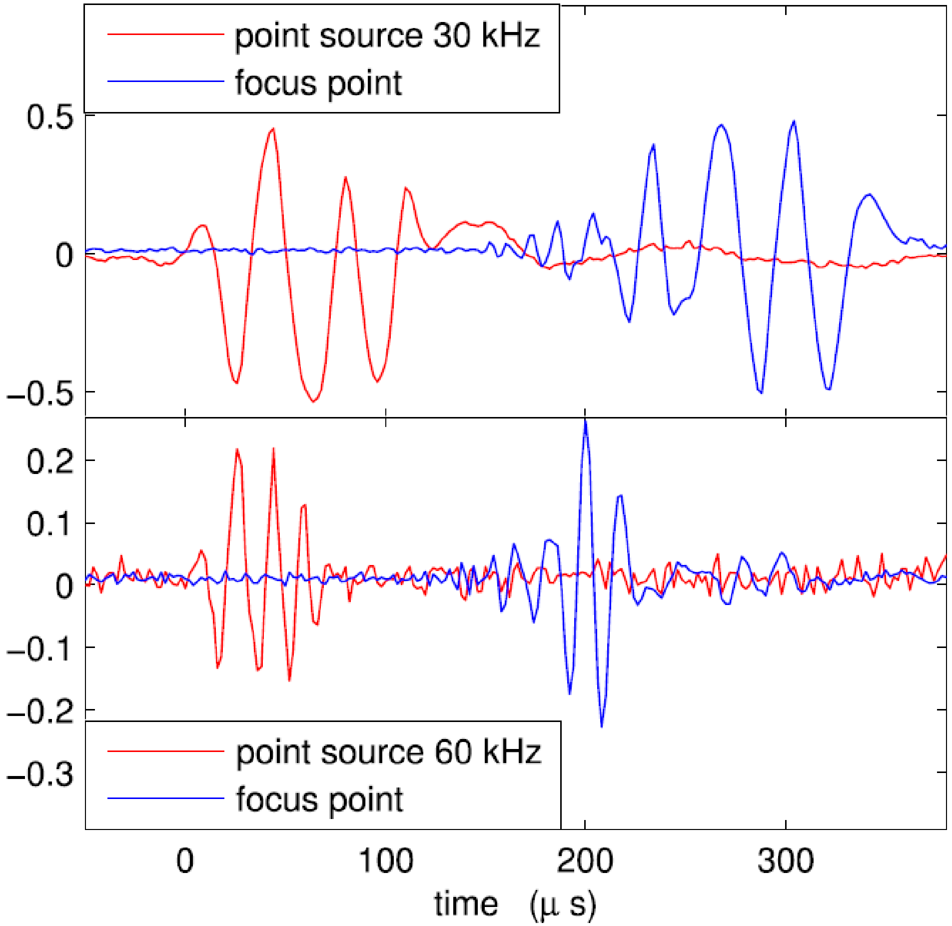


60 kHz wavepacket

Dispersion in Plates: Isotropic Case



30 kHz wavepacket



Experiments on Maxwell's fish-eye dynamics in elastic plates, Applied. Phys. Lett. 2015

Dispersion in Plates: Orthotropic Case

Flexural waves equation in an orthotropic plate:

$$\rho h \frac{\partial^2 u}{\partial t^2} + D_x \frac{\partial^4 u}{\partial x^4} + D_y \frac{\partial^4 u}{\partial y^4} + 2 D_{xy} \frac{\partial^4 u}{\partial x^2 \partial y^2} = 0$$

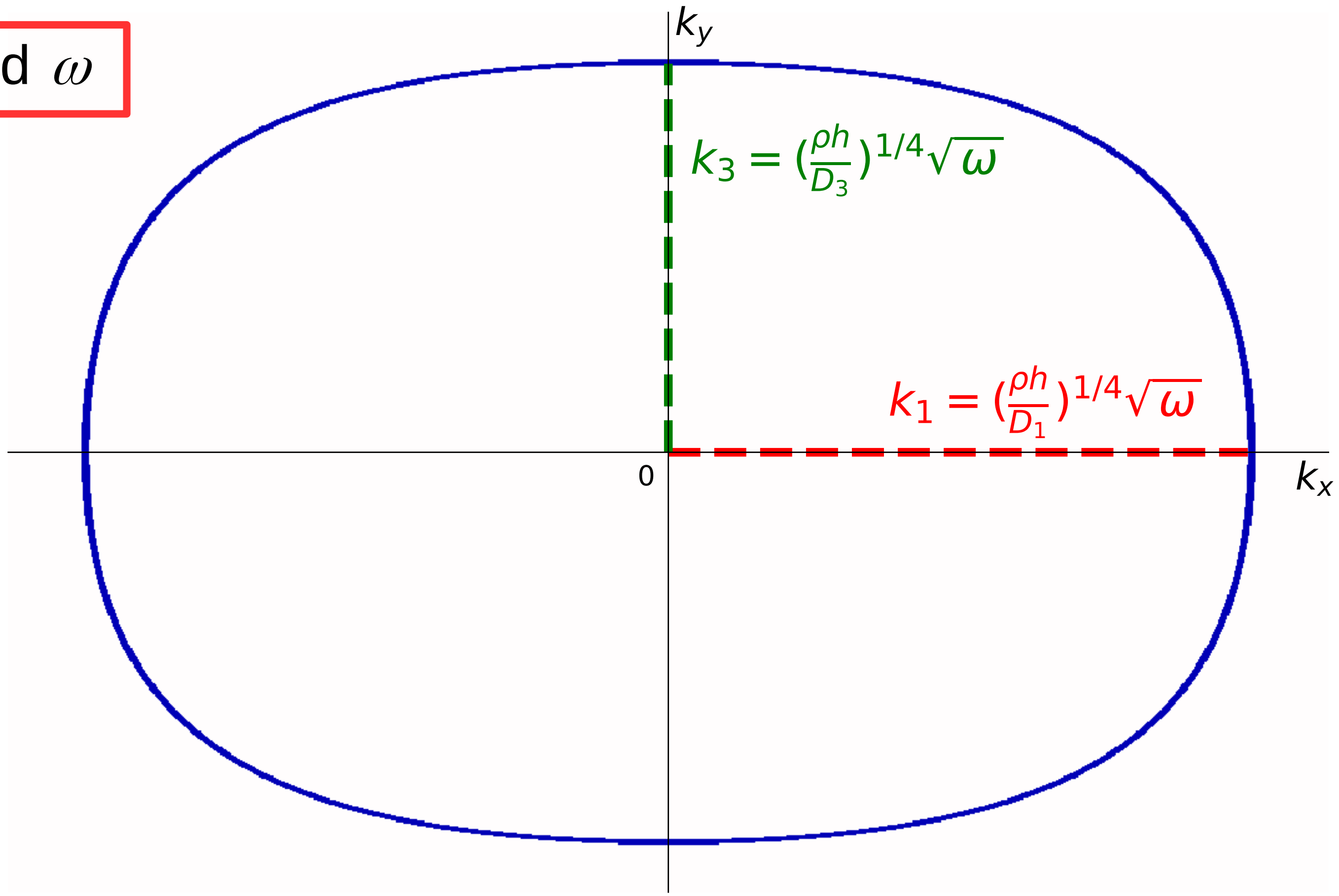
Dispersion relation:

$$\rho h \omega^2 = D_1 k_x^4 + D_3 k_y^4 + 2 D_{xy} k_x^2 k_y^2$$

For fixed ω, equation of an ellipse with variables k_x^2, k_y^2

Dispersion in Plates: Orthotropic Case

Fixed ω



$$\rho h \omega^2 = D_x k_x^4 + D_y k_y^4 + 2 D_{xy} k_x^2 k_y^2$$

- Wave speed depends on the direction of propagation
- Wavenumber growing with frequency

Dispersion in Plates: Orthotropic Case

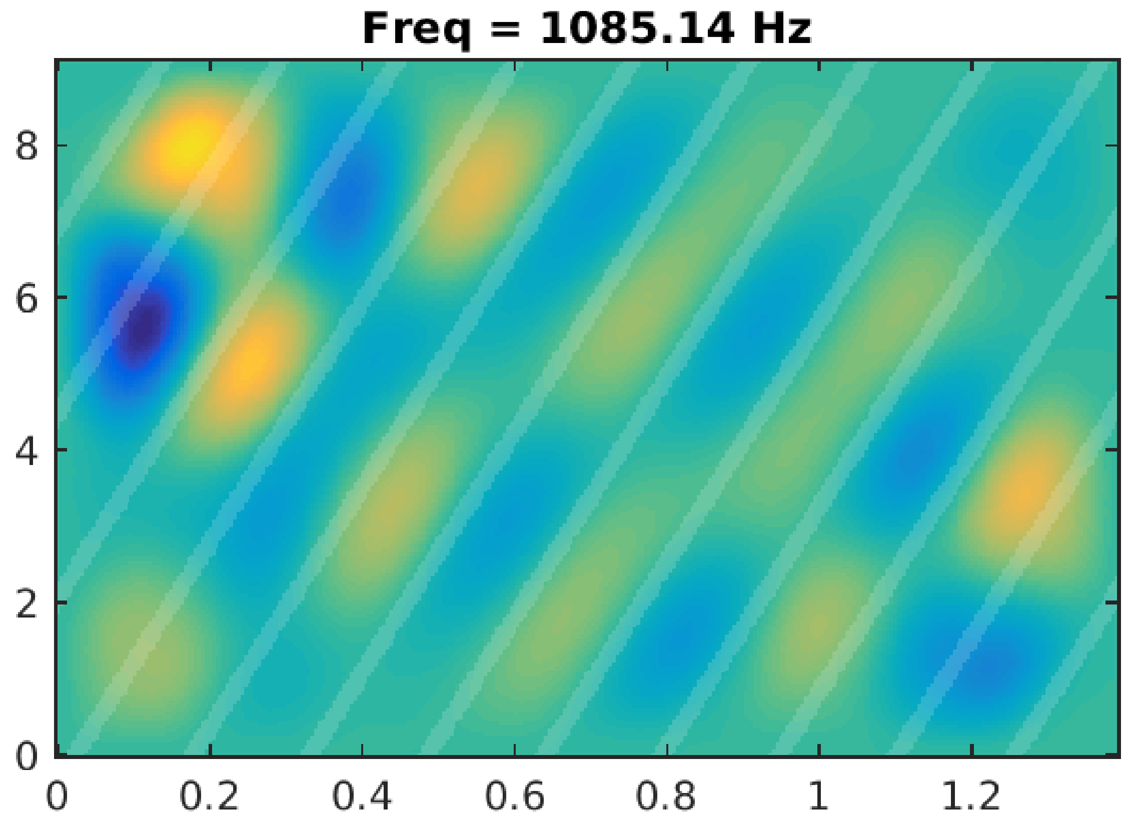
- Orthotropic materials are common, e.g.: wood, composites with fibres (carbon, glass)

- Only 4 coefficients needed to characterize the plate flexion

	E_x (GPa)	E_y (GPa)	μ (GPa)	ν
Norway Spruce	15.8	0.85	0.84	0.3
Sitka Spruce	11.5	0.47	0.5	0.3
Fir	8.86	0.54	1.6	0.3
Maple	10	2.2	2	0.3

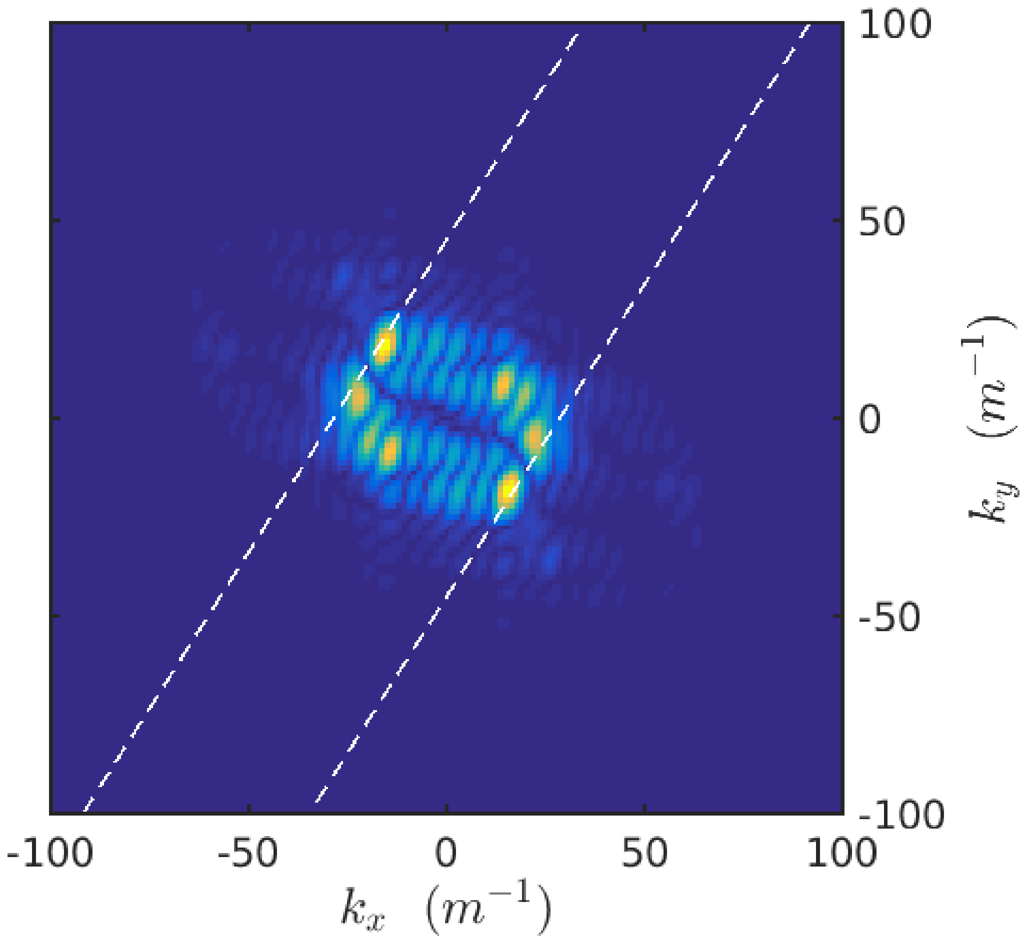
Ribbed Plates

Structuration brings anisotropy



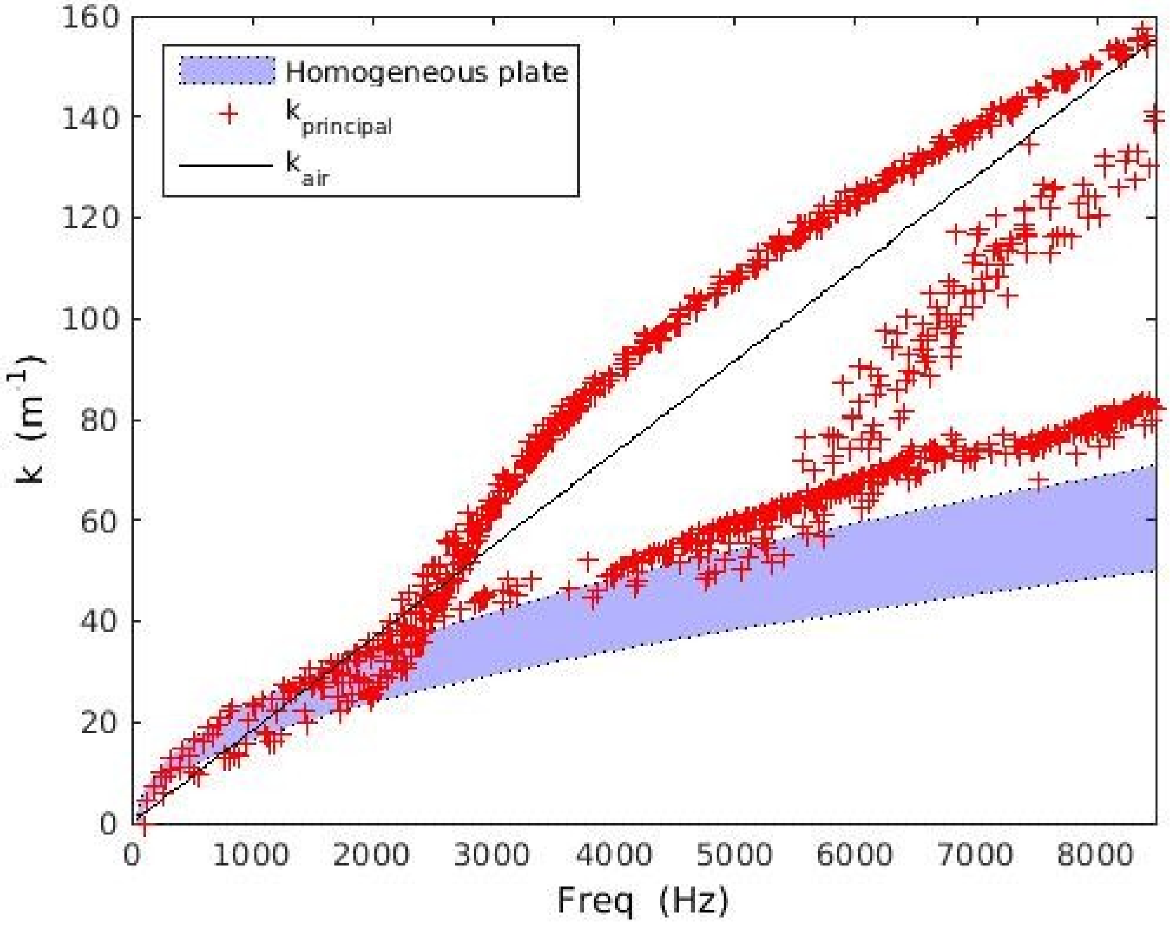
Ribbed plate

2D Fourier Transform (central symmetry $k \to -k$)



Ribbed Plates

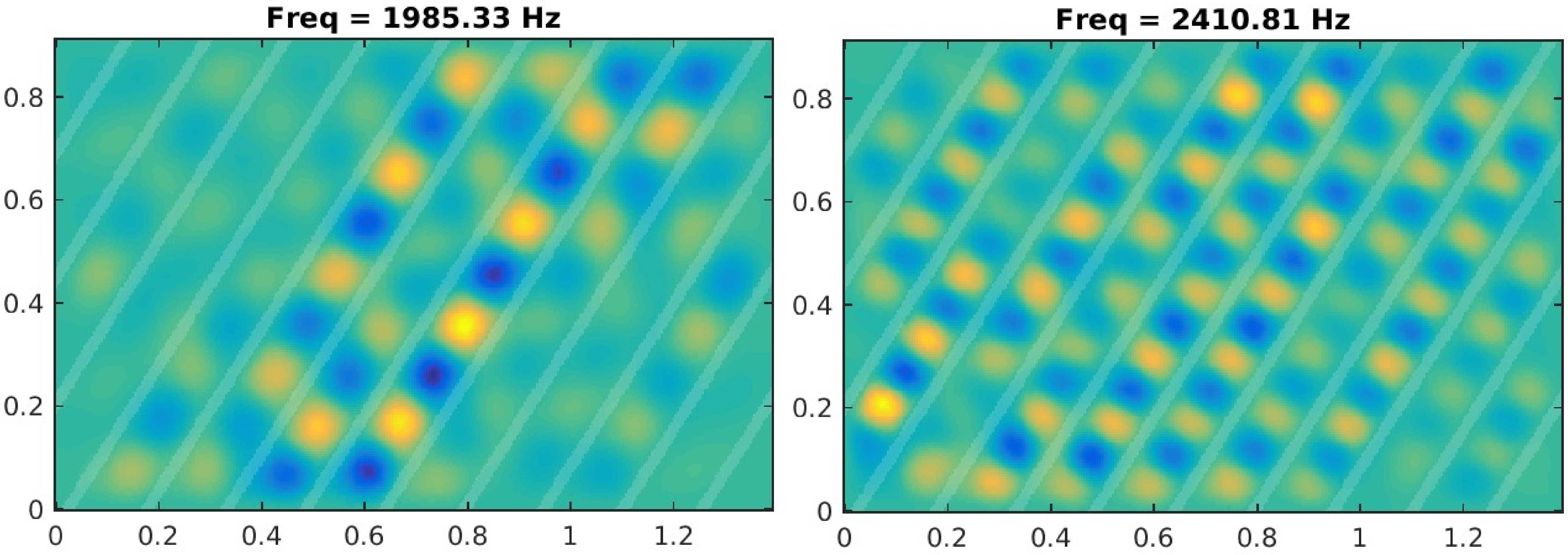
2D Fourier transform allows to obtain experimental dispersion



$k_{principal}$: dominant wavenumber of the mode

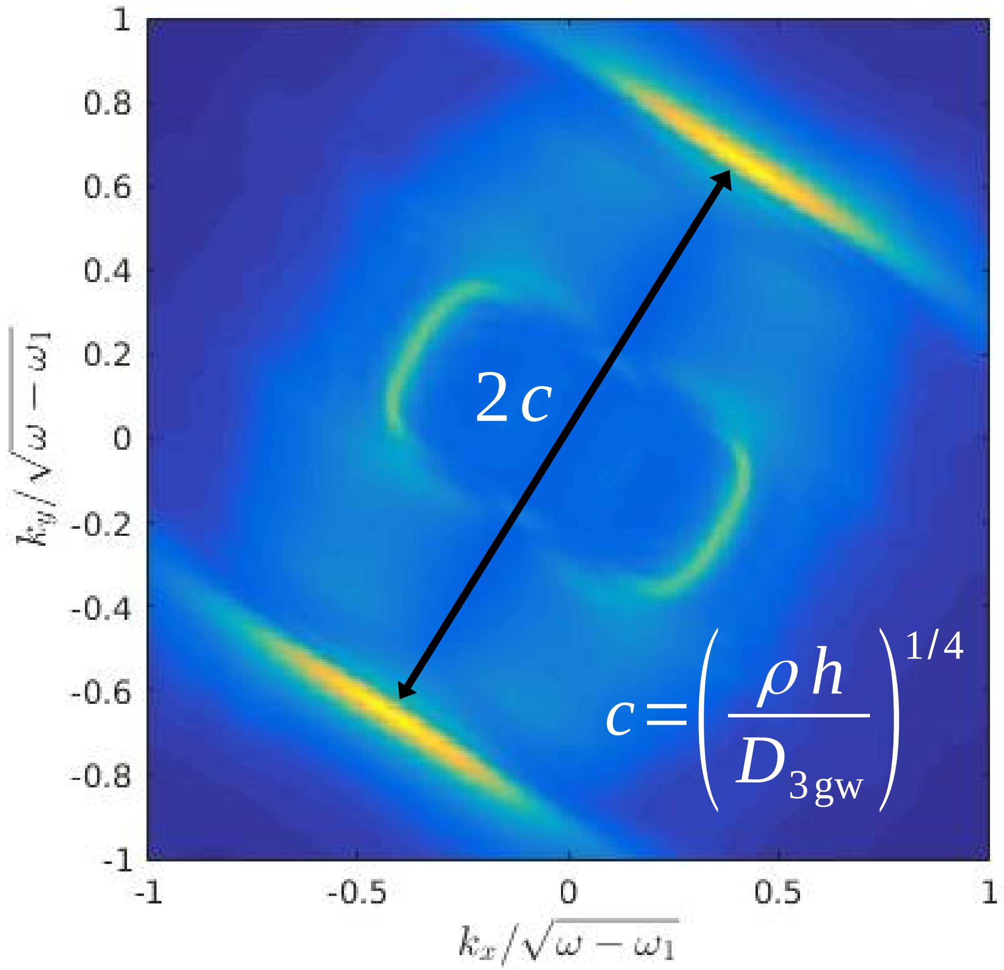
- Dispersion identical to the homogeneous plate up to 2kHz
- Wavenumbers are increased by the presence of bars
- Additional dispersion branches appear

Ribbed Plates



Modes_2D-FFT

Ribbed Plates



Spatial dispersion map rescaled by $\sqrt{\varpi}$

Spatial dispersion map rescaled by $\sqrt{\varpi-\varpi_1}$

Periodic Structures

Periodically pinned plate

Dispersion maps can be obtained by Floquet-Bloch theory, or homogenisation methods

Periodic Structures

Extreme anisotropy effects

Unveiling Extreme Anisotropy in Structured Media, Phys. Rev. Lett. 2016

95 kHz 110 kHz 119 kHz