Elastic waves in solids Crash Course

Gautier Lefebvre

Université de Technologie de Compiègne

ALLIANCE SORBONNE UNIVERSITÉ | utc Recherche Roberval

Roberval laboratory

Outline of the Lecture

I. Elastic Wave Propagation in Free Space

1. 3D Elasticity
2. Longitudinal and Transverse Waves

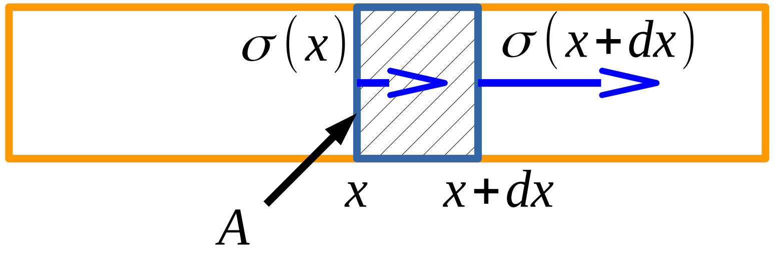
II. Reflection and transmission through interfaces

1. Boundary Conditions
2. Free Surface Reflection
3. Solid-Fluid Interface

III. Guided Waves

1. Surface Waves
2. Lamb Waves

Elastic Waves in Solids: 1D Solid



1D medium along x, independent of y or z.

$$dm\frac{\partial^2 u}{\partial t^2}=\sigma(x+dx)A-\sigma(x)A \quad dm=\rho A\,dx$$

$$\rho A\frac{\partial^2 u}{\partial t^2}=\frac{\sigma(x+dx)-\sigma(x)}{dx}A=\frac{\partial\sigma}{\partial x}A$$

Young's modulus

$$\sigma=E\varepsilon \quad \textit{Hooke's Law}$$

$$\rho\frac{\partial^2 u}{\partial t^2}=\frac{\partial\sigma}{\partial x}=\frac{\partial(E\varepsilon)}{\partial x}=E\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$$

$$\frac{\partial^2 u}{\partial t^2}=\frac{E}{\rho}\frac{\partial^2 u}{\partial x^2}$$

3D solid medium

$\vec{u}(\vec{x})$: displacement from equilibrium at position x. $\vec{x}=(x_1, x_2, x_3)$

$$\vec{u}(\vec{x},t)=(u_1(\vec{x},t), u_2(\vec{x},t), u_3(\vec{x},t))$$

$$u_i(\vec{x}+d\vec{x})=u_i(\vec{x})+\frac{\partial u_i}{\partial x_j}dx_j$$

No deformation if the gradient of displacement is zero.

$$du_i(\vec{x}+d\vec{x})=\frac{\partial u_i}{\partial x_j}dx_j=\underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)}_{\textit{Symmetric}}dx_j+\underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}-\frac{\partial u_j}{\partial x_i}\right)}_{\textit{Antisymmetric}}dx_j$$

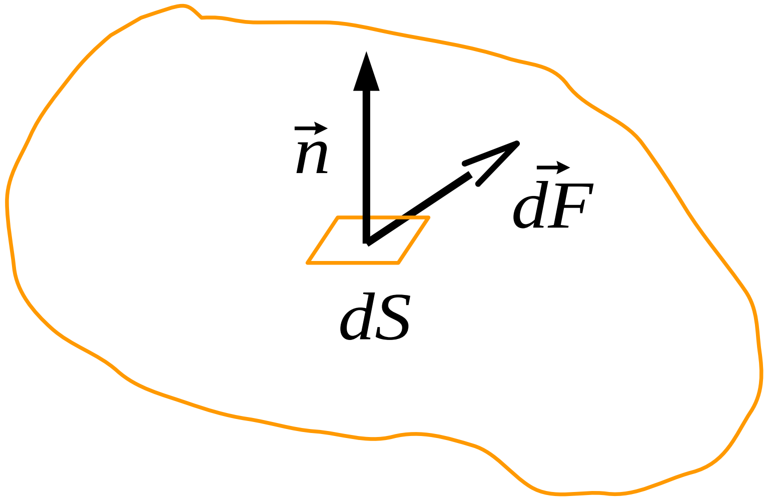
Strain tensor $\varepsilon_{ij}=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)$

$\omega_{ij}=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}-\frac{\partial u_j}{\partial x_i}\right)$ Describe a rotation movement

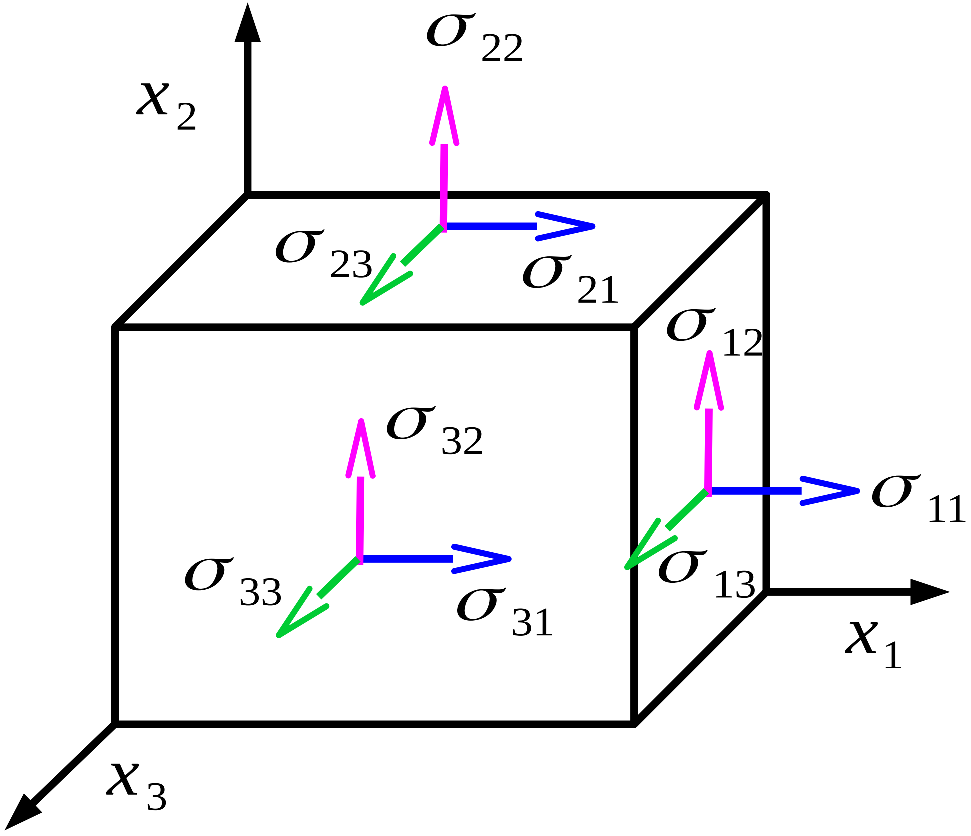
→ no vibrations

Stress Tensor

$$\vec{dF} = \vec{n}\,\underline{\sigma}\,dS$$

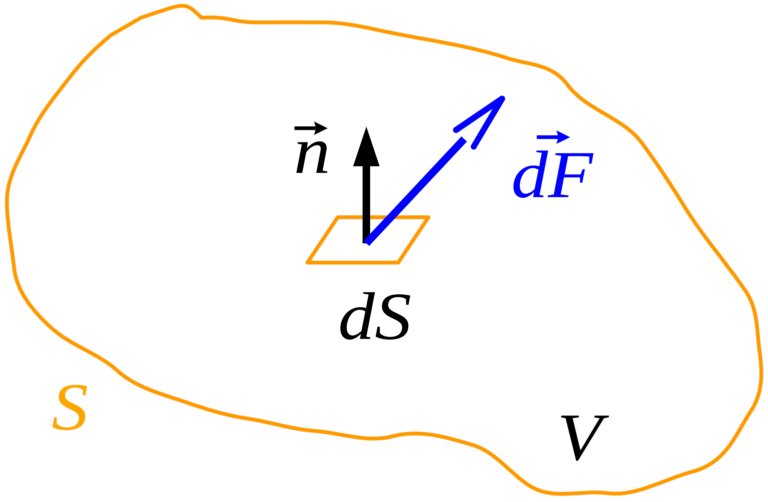


$$\underline{\sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}$$



Elastic wave equation

$$d\vec{F} = \vec{n}\,\underline{\sigma}\,dS$$



The total stress force acting on the body of volume V is:

$$\iint_S \vec{n}\,\underline{\sigma}\,dS = \iiint_V \mathrm{div}\,\underline{\sigma}\,dV$$

Gauss theorem

Balance of forces:

$$\iiint_V \rho \frac{\partial^2 \vec{u}}{\partial t^2} dV = \iiint_V \mathrm{div}\,\underline{\sigma}\,dV + \iiint_V \vec{f}\,dV$$

with f is the density of external forces

$$\rho \frac{\partial^2 \vec{u}}{\partial t^2} = \mathrm{div}\,\underline{\sigma} + \vec{f} \qquad \rho \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial \sigma_{ij}}{\partial x_j} + f_i$$

Symmetry of the stress tensor

When there is no density of moment, we can show that the stress tensor is symmetric:

$$\sigma_{ij} = \sigma_{ji} \qquad i, j = 1,2,3$$

Only 6 independent components:

- 3 Normal stresses $\sigma_{11}, \sigma_{22}, \sigma_{33}$
- 3 Tangential stresses $\sigma_{12} = \sigma_{21}, \sigma_{13} = \sigma_{31}, \sigma_{23} = \sigma_{32}$

Linear Elasticity

Assuming a linear link between stress tensor and deformation

$$\sigma_{ij} = c_{ijkl}\,\varepsilon_{kl}$$

c_{ijkl} : Elasticity tensor, fourth-rank ($3^4 = 81$ terms)

From the definition, we have the same symmetries as σ_{ij} and ε_{kl} :

$$c_{ijkl} = c_{jikl} = c_{ijlk}$$

By considering a reversible transformation, we can also show:

$$c_{ijkl} = c_{klij}$$

81 terms $\rightarrow$ 21 terms

$$\sigma_{ij} = c_{ijkl}\,\varepsilon_{kl} = \frac{1}{2} c_{ijkl} \left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k} \right) = c_{ijkl} \frac{\partial u_k}{\partial x_l}$$

Elasticity Notations

Voigt notation using the fundamental symmetries

of σ and ε:

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{Bmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ c_{21} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ c_{31} & c_{32} & c_{33} & c_{34} & c_{35} & c_{36} \\ c_{41} & c_{42} & c_{43} & c_{44} & c_{45} & c_{46} \\ c_{51} & c_{52} & c_{53} & c_{54} & c_{55} & c_{56} \\ c_{61} & c_{62} & c_{63} & c_{64} & c_{65} & c_{66} \end{pmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{23} \\ \varepsilon_{31} \\ \varepsilon_{12} \end{Bmatrix}$$

- Isotropic: 2 coefficients
- Axi-symetrical: 5 coefficients (Glass wool)
- Orthotropic: 9 coefficients (Wood)

Most general notation, for any kind of material

Isotropic Medium

For an isotropic medium, the elasticity tensor can be written as:

$$c_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})$$

Related to compression

with the Lamé coefficients λ and μ *Related to shear*

$$\sigma_{ij} = c_{ijkl} \varepsilon_{kl} = \lambda \delta_{ij} \delta_{kl} \varepsilon_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \varepsilon_{kl}$$

$$\sigma_{ij} = \lambda \varepsilon_{kk} + 2 \mu \varepsilon_{ij} \qquad \text{or} \qquad \underline{\sigma} = \lambda \, \underline{\mathrm{Id}} \, \mathrm{Tr} \, \underline{\varepsilon} + 2 \mu \underline{\varepsilon}$$

In terms of u, we can write:

$$\vec{\mathrm{div}} \, \underline{\sigma} = (\lambda + \mu) \vec{\mathrm{grad}} (\mathrm{div} \, \vec{u}) + \mu \Delta \vec{u}$$

Hooke's Law

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{23} \\ \varepsilon_{31} \\ \varepsilon_{12} \end{Bmatrix} = \frac{1}{E} \begin{pmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1+\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1+\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1+\nu \end{pmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{Bmatrix}$$

Suitable for isotropic materials (2 parameters only)

Possibility to link the different notations, e.g.:

$$\lambda = \frac{\nu E}{(1-2\nu)(1+\nu)} = c_{11} - 2c_{44}, \qquad \mu = \frac{E}{2(1+\nu)} = c_{44}$$

Conversion formula

Uncoupling of the Wave Equation

Wave equation:

$$\rho \frac{\partial^2 \vec{u}}{\partial t^2} = \vec{\mathrm{div}}\, \underline{\sigma} + \vec{f}$$

$$\rho \frac{\partial^2 \vec{u}}{\partial t^2} = (\lambda + \mu)\, \vec{\mathrm{grad}}\,(\mathrm{div}\, \vec{u}) + \mu\, \vec{\Delta}\, \vec{u} + \vec{f}$$

Laplacian can be written as: $\vec{\Delta}\, \vec{u} = \vec{\mathrm{grad}}\,(\mathrm{div}\, \vec{A}) - \vec{\mathrm{rot}}\,(\vec{\mathrm{rot}}\, \vec{A})$

$$\rho \frac{\partial^2 \vec{u}}{\partial t^2} = (\lambda + 2\mu)\, \vec{\mathrm{grad}}\,(\mathrm{div}\, \vec{u}) - \mu\, \vec{\mathrm{rot}}\,(\vec{\mathrm{rot}}\, \vec{u}) + \vec{f}$$

Helmholtz Decomposition

We write the displacement field as the sum of two terms:

$$\vec{u} = \vec{u}_L + \vec{u}_T \qquad \begin{aligned} \vec{u}_L &= \overrightarrow{\text{grad}}\,\phi \\ \vec{u}_T &= \overrightarrow{\text{rot}}\,\vec{\psi} \end{aligned}$$

ϕ is a scalar potential, $\vec{\psi}$ a vector potential.

$$\overrightarrow{\text{rot}}\,\vec{u}_L = \vec{0}\,, \qquad \text{div}\,\vec{u}_T = 0$$

Thus, we can separate the two components:

$$\begin{cases} \dfrac{\partial^2 \vec{u}_L}{\partial t^2} - c_L\,\overrightarrow{\text{grad}}\left(\text{div}\,\vec{u}_L\right) = 0\,, & c_L = \sqrt{\dfrac{\lambda + 2\mu}{\rho}} \\ \dfrac{\partial^2 \vec{u}_L}{\partial t^2} + c_T^2\,\overrightarrow{\text{rot}}\left(\overrightarrow{\text{rot}}\,\vec{u}_T\right) = 0\,, & c_T = \sqrt{\dfrac{\mu}{\rho}} \end{cases}$$

Helmholtz Decomposition

We obtain d'Alembert equation for displacements and potential:

$$\begin{cases} \dfrac{\partial^2 \vec{u}_L}{\partial t^2} - c_L \vec{\Delta} \vec{u}_L = \vec{0}, & \dfrac{\partial^2 \phi}{\partial t^2} - c_L \Delta \phi = 0 \\ \dfrac{\partial^2 \vec{u}_T}{\partial t^2} - c_T^2 \vec{\Delta} \vec{u}_T = \vec{0}, & \dfrac{\partial^2 \vec{\psi}}{\partial t^2} - c_T^2 \vec{\Delta} \vec{\psi} = \vec{0} \end{cases}$$

We replaced 3 unknown displacement components by 4 potential components (1 for scalar, 3 for vectorial). A gauge equation might be needed:

$$\operatorname{div} \vec{\psi} = 0$$

Examples of Material Properties

Material	ρ (g/cm^3)	λ (GPa)	μ (GPa)	c_L (mm/µs)	c_T (mm/µs)
Steel	7.5-9	95-140	75-90	5.2-6.5	2.9-3.5
Aluminium	2.7	58	26	6.3	3.1
Concrete	1.3-2.5	5-14	8-21	2.9-6.5	1.8-4
Diamond	3.5	340	510	19.7	12.1
Epoxy	1.1-1.4	1.4-3.6	0.7-1.9	1.4-2.6	0.7-1.3
Plexiglass	1.2	3.4-7.8	0.8-1.2	2-2.9	0.8-1
Lead	11.5	39	6.3	2.1	0.75
PVC	1.1-1.5	1.7-5.7	0.7-1.5	1.4-2.8	0.7-1.2
Titanium	4.5	77	43	6	3
Glass	2.4-2.8	14-25	20-38	4.4-6.5	2.7-4

Plane wave solutions

As for acoustic waves, plane wave are solutions for infinite space:

$$\vec{u}(\vec{x},t)=A\vec{P}e^{(\vec{k}\vec{x}-\omega t)}$$

A is the amplitude, $\vec{P}$ a polarisation vector depending on $\vec{n}$ as $\vec{k}=k\vec{n}$.

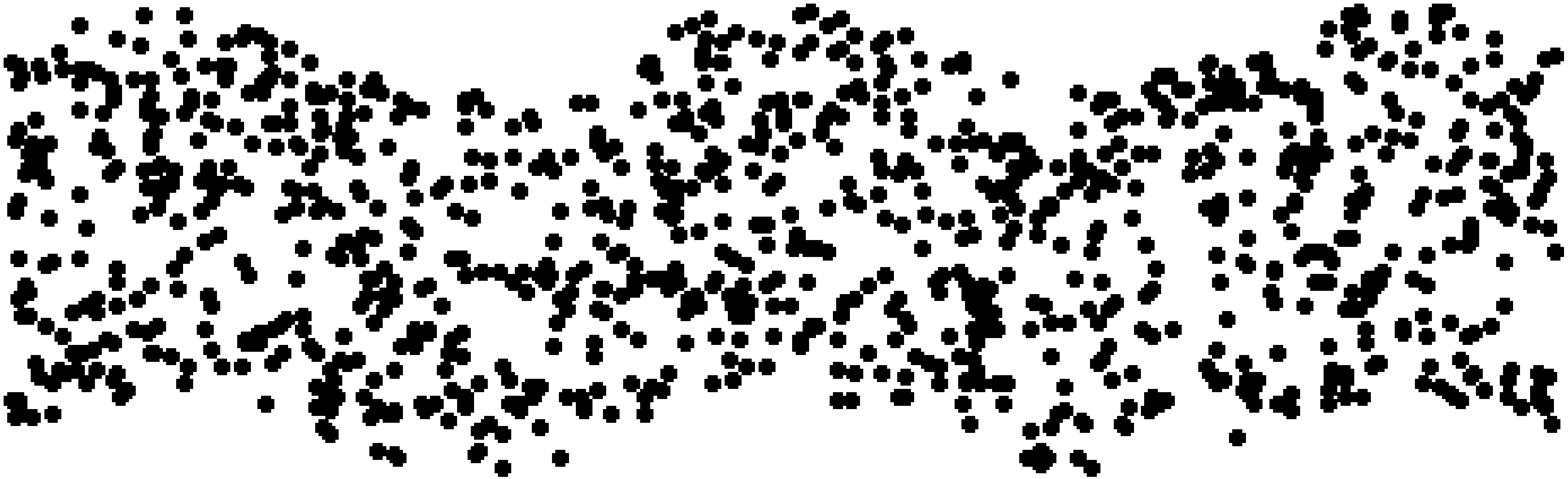
For longitudinal waves: $\vec{\mathrm{rot}}\,\vec{u}_L=\vec{0} \quad \Rightarrow \quad \vec{n}\wedge\vec{P}_L=\vec{0}$

For transverse waves: $\mathrm{div}\,\vec{u}_T=0 \quad \Rightarrow \quad \vec{n}.\vec{P}_T=0$

Animations

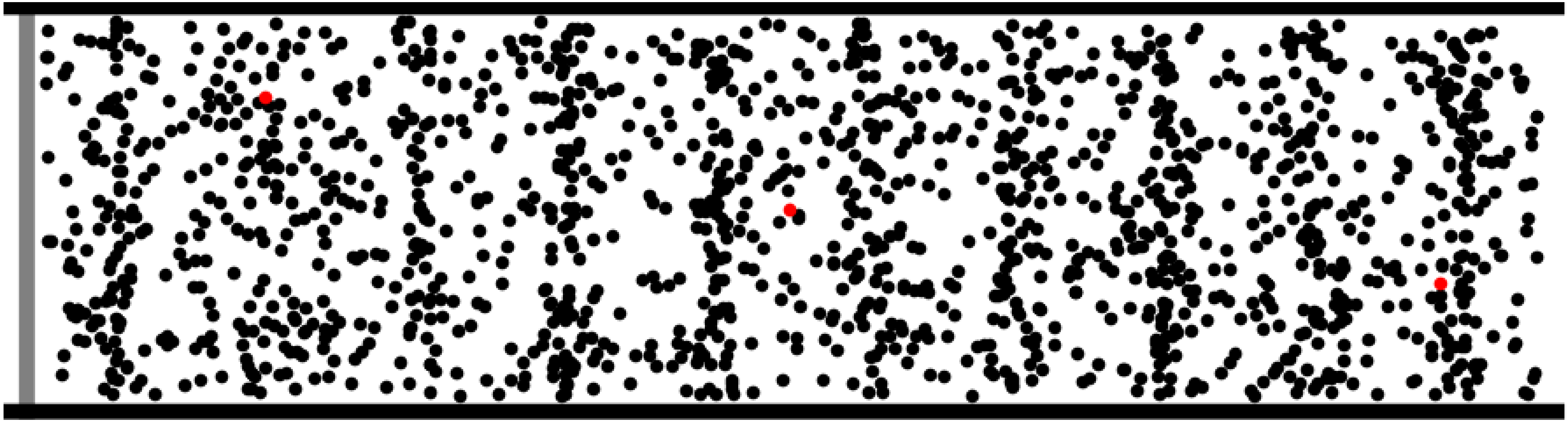
Example: $\vec{n}=\vec{e}_1 \quad \Rightarrow \quad \vec{P}_L=\vec{e}_1, \quad \vec{P}_{T_1}=\vec{e}_2$ and $\vec{P}_{T_2}=\vec{e}_3$

Transverse Waves



https://www.acs.psu.edu/drussell/demos.html

Longitudinal Waves



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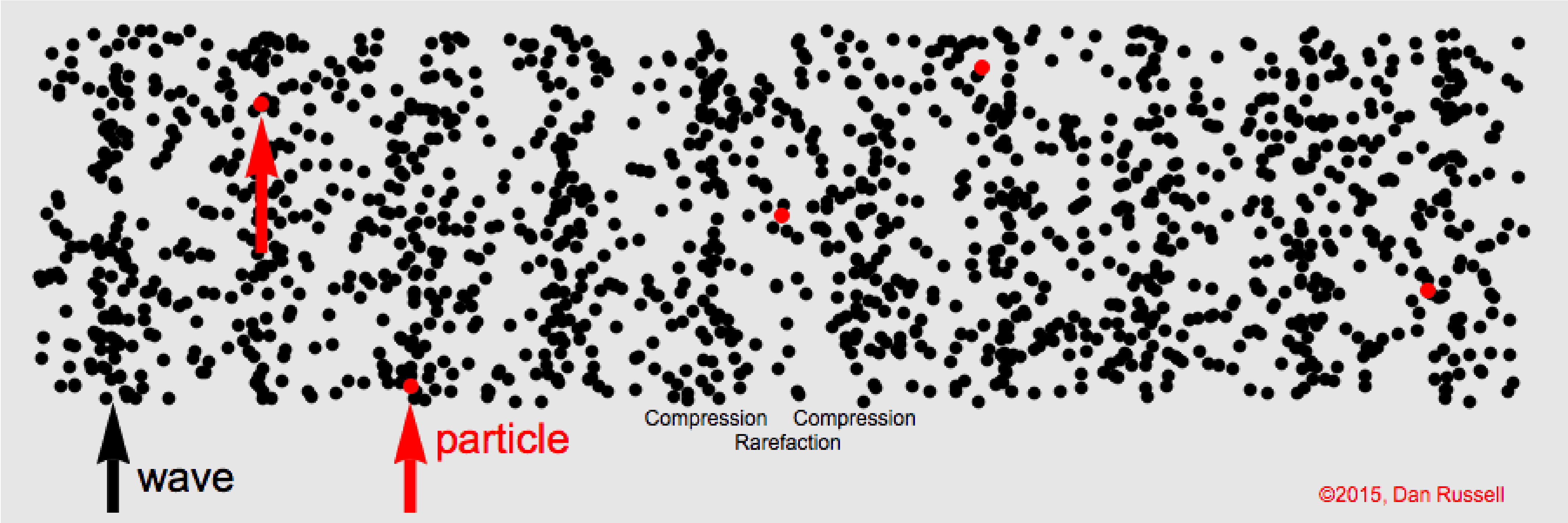
Longitudinal Waves



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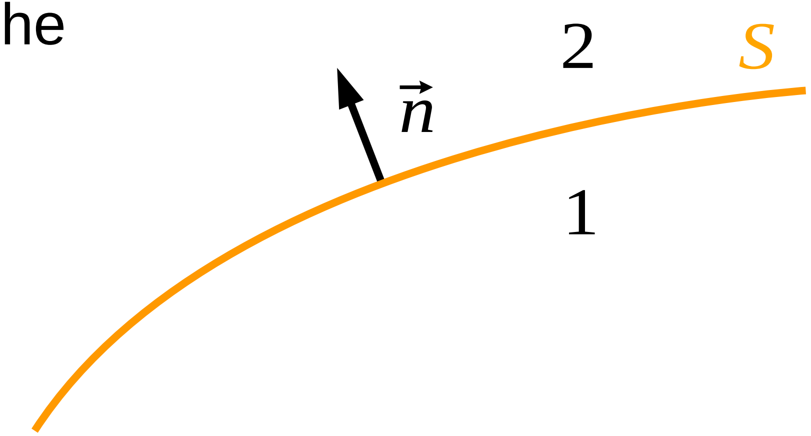
Boundary Conditions

Let's consider a surface S between two solids, normal $\vec{n}$. The continuity conditions for any points in S write:



$$\begin{cases} \underline{\sigma_1}(\vec{x},t)\vec{n} = \underline{\sigma_2}(\vec{x},t)\vec{n} \\ \vec{u}_1(\vec{x},t) = \vec{u}_2(\vec{x},t) \end{cases}$$

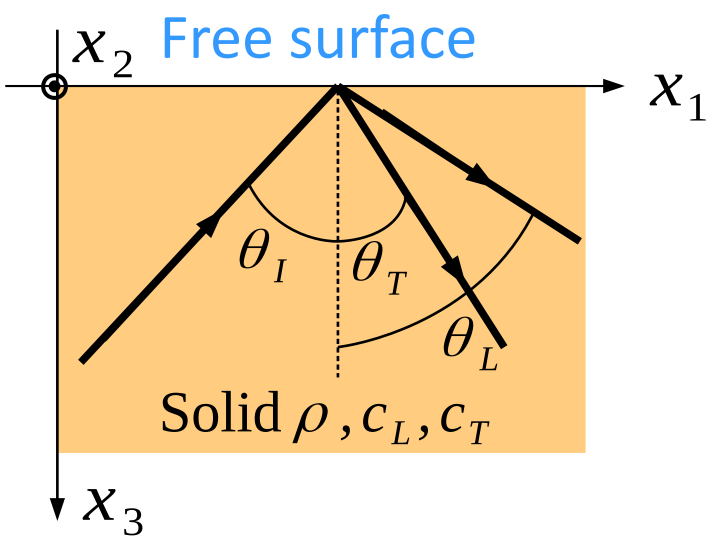
If the medium 2 is a perfect fluid, the stress tensor is: $\underline{\sigma_2} = -P_2 \underline{\mathrm{Id}}$

$$\begin{cases} \underline{\sigma_1}(\vec{x},t)\vec{n} = -P_2(\vec{x},t)\vec{n} \\ \vec{u}_1(\vec{x},t).\vec{n} = \vec{u}_2(\vec{x},t).\vec{n} \end{cases}$$

No shear stress in a fluid, and only normal displacement continuity.

Free-surface reflection

Reflection on a free space: $\underline{\sigma}(\vec{x},t).\vec{e}_3=\vec{0}$



Incident plane wave: $\vec{u}_I(\vec{x},t)=A_I\vec{P}_I e^{(k\vec{n}_I\vec{x}-\omega t)}$

$$\vec{P}=\begin{pmatrix}P_1\\P_2\\P_3\end{pmatrix},\ \text{with}\ \ \vec{n}_I=\begin{pmatrix}\sin\theta_I\\0\\\cos\theta_I\end{pmatrix},\ \text{and}\ \ k_I=\omega/c_I$$

For each point in S,

Snell-Descartes Law

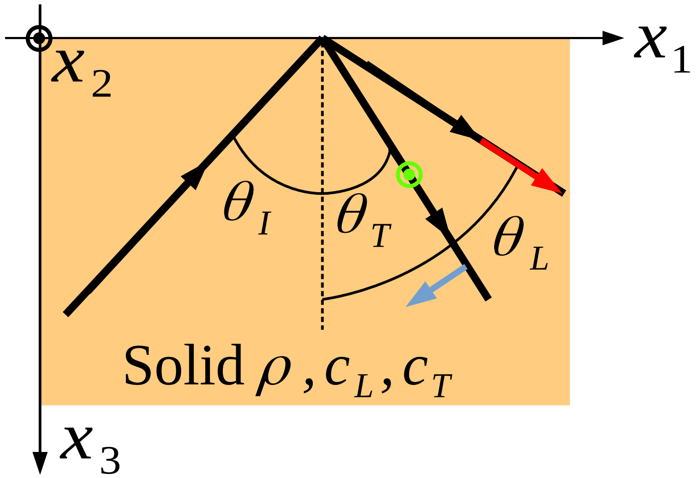
$$\vec{k}_I.\vec{x}=\vec{k}_L.\vec{x}=\vec{k}_T.\vec{x},\quad \text{then}\quad \frac{\sin\theta_I}{c_I}=\frac{\sin\theta_L}{c_L}=\frac{\sin\theta_T}{c_T}$$

Free-surface reflection

Reflected waves:

$$\vec{u}_R(x_1,x_3,t)=\vec{u}_L(x_1,x_3,t)+\vec{u}_{T_1}(x_1,x_3,t)+\vec{u}_{T_2}(x_1,x_3,t)$$

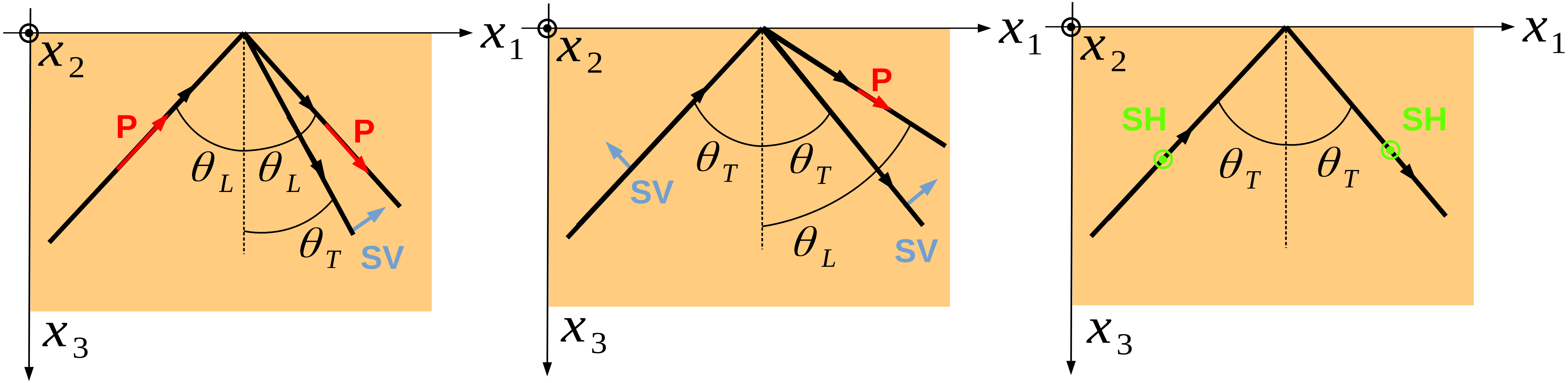
$$\begin{cases} \vec{u}_L(\vec{x},t)=A_L\vec{P}_L e^{(\vec{k}_l.\vec{x}-\omega t)} \\ \vec{u}_{T_1}(\vec{x},t)=A_{T_1}\vec{P}_{T_1} e^{(\vec{k}_T.\vec{x}-\omega t)} \\ \vec{u}_{T_2}(\vec{x},t)=A_{T_2}\vec{P}_{T_2} e^{(\vec{k}_T.\vec{x}-\omega t)} \end{cases}$$



Using: $\vec{k}_L \wedge \vec{P}_L=\vec{0}$, $\vec{k}_T.\vec{P}_T=0$

P $\vec{P}_L=\dfrac{c_L}{\omega}\begin{pmatrix} k_1 \\ 0 \\ -k_{3L} \end{pmatrix}$, **SV** $\vec{P}_{T_1}=\dfrac{c_T}{\omega}\begin{pmatrix} k_{3T} \\ 0 \\ -k_1 \end{pmatrix}$, **SH** $\vec{P}_{T_2}=\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

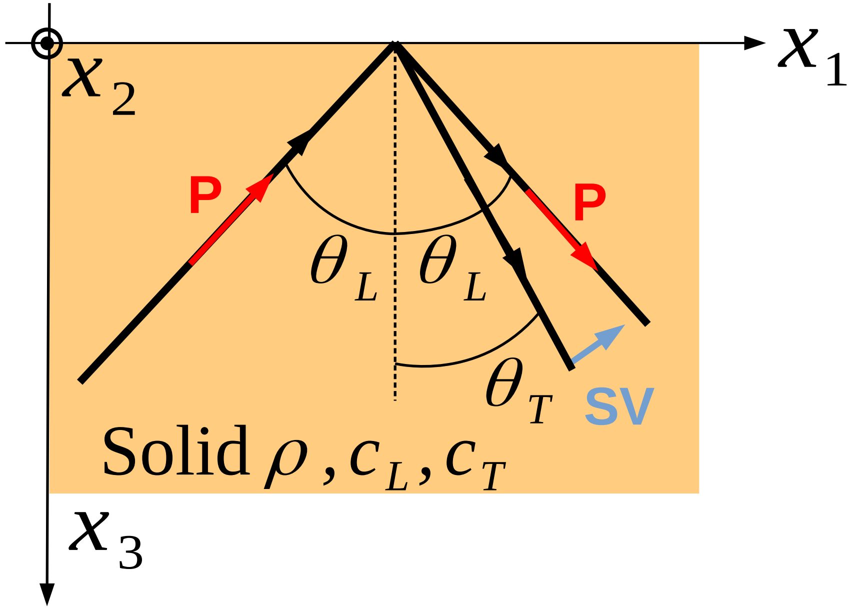
Free-surface reflection



Incident Longitudinal Wave

Using the free surface boundary condition $\underline{\sigma}(\vec{x},t).\vec{e}_3=\vec{0}$,

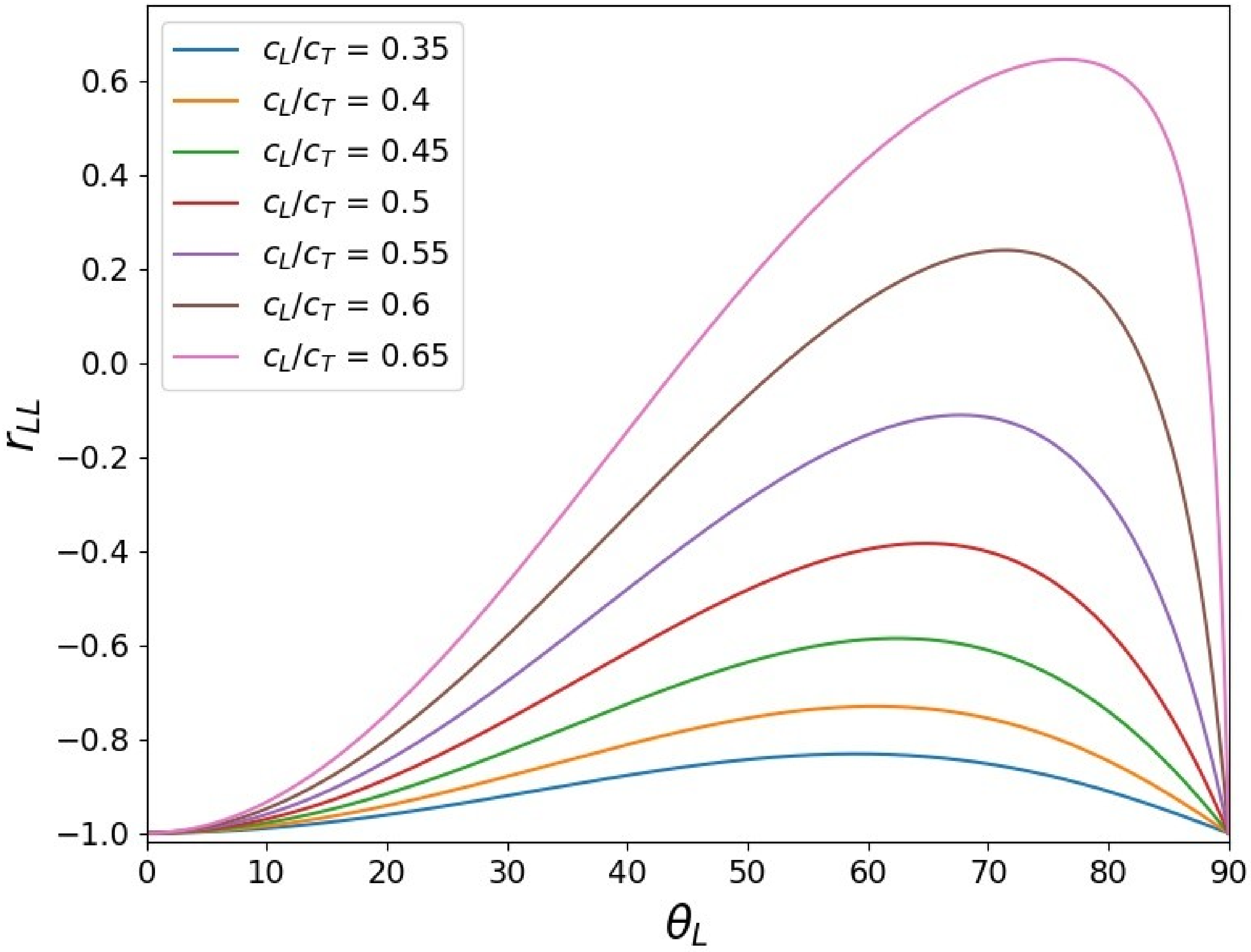
we can obtain the reflection coefficients:



$$r_{LL}=\frac{A_L}{A_I}=\frac{\left(\frac{c_T}{c_L}\right)^2\sin 2\theta_L \sin 2\theta_T-\cos^2 2\theta_T}{\left(\frac{c_T}{c_L}\right)^2\sin 2\theta_L \sin 2\theta_T+\cos^2 2\theta_T},$$

$$r_{LT}=\frac{A_T}{A_I}=\frac{2\left(\frac{c_T}{c_L}\right)\sin 2\theta_L \cos 2\theta_T}{\left(\frac{c_T}{c_L}\right)^2\sin 2\theta_L \sin 2\theta_T+\cos^2 2\theta_T}, \quad \text{with} \quad \sin\theta_T=\frac{c_T}{c_L}\sin\theta_L$$

Incident Longitudinal Wave



- The amplitude is maximal for normal incidence or grazing incidence (no reflected shear wave).

- The conversion to shear waves can be perfect for specific angles as $r_{LL}=0$, if $\frac{c_L}{c_T}>0.565$

Incident Transverse Vertical Wave

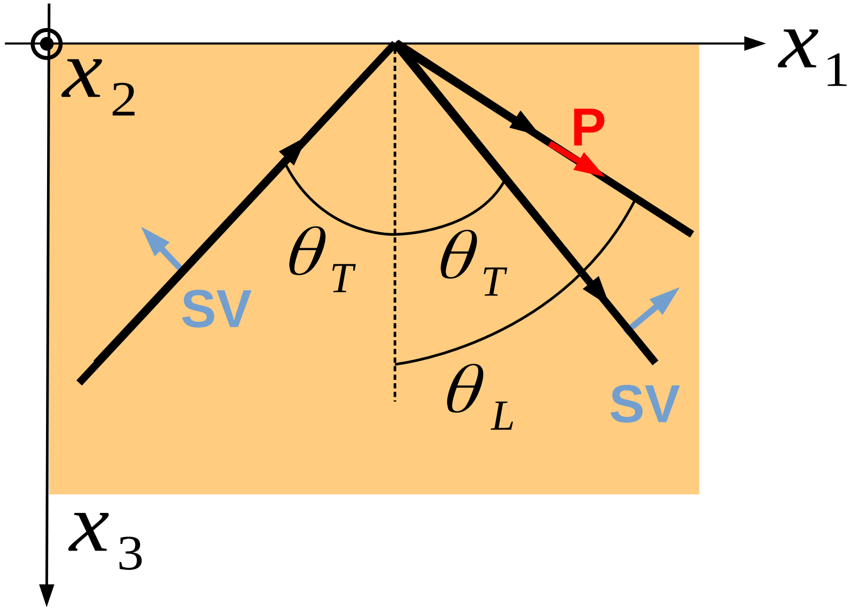
$$\sin\theta_L = \frac{c_L}{c_T}\sin\theta_T$$

As shear waves are usually slower than compressive waves, $\theta_L > \theta_T$.

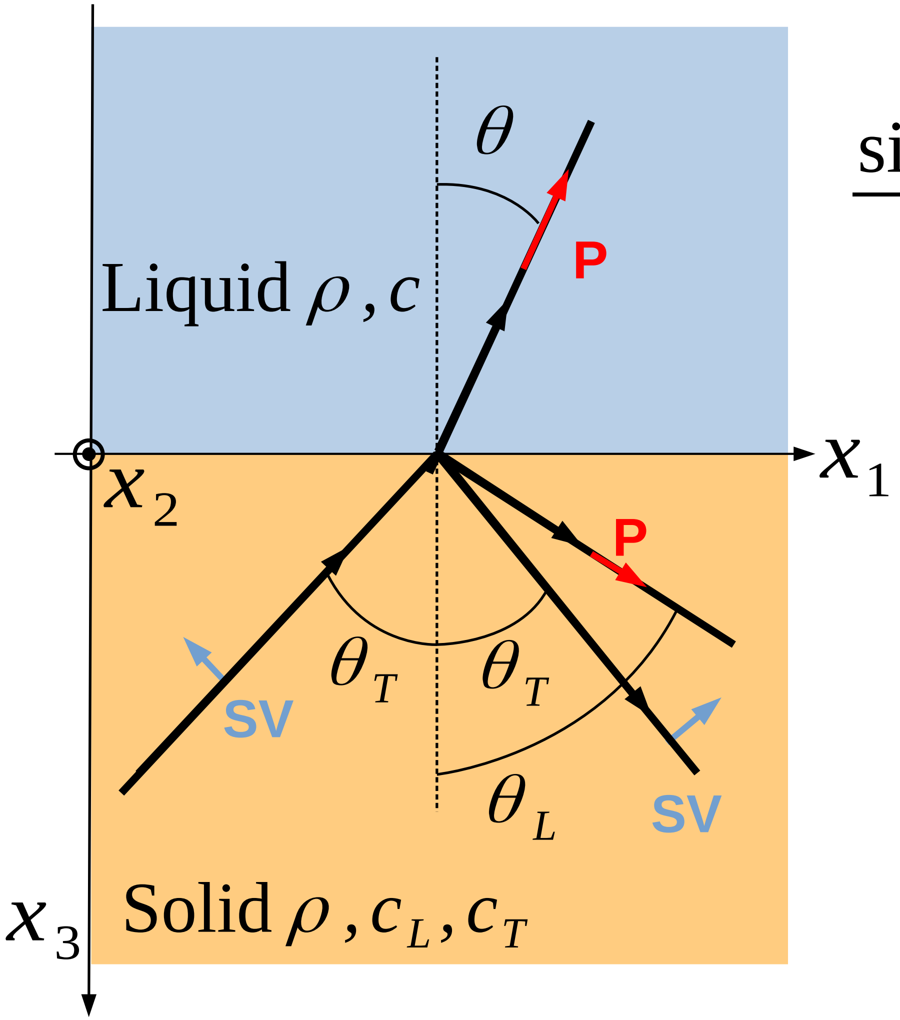
A critical incident angle is reached as $\theta_L = 90°$,

$$\theta_T^c = \arcsin\left(\frac{c_T}{c_L}\right)$$

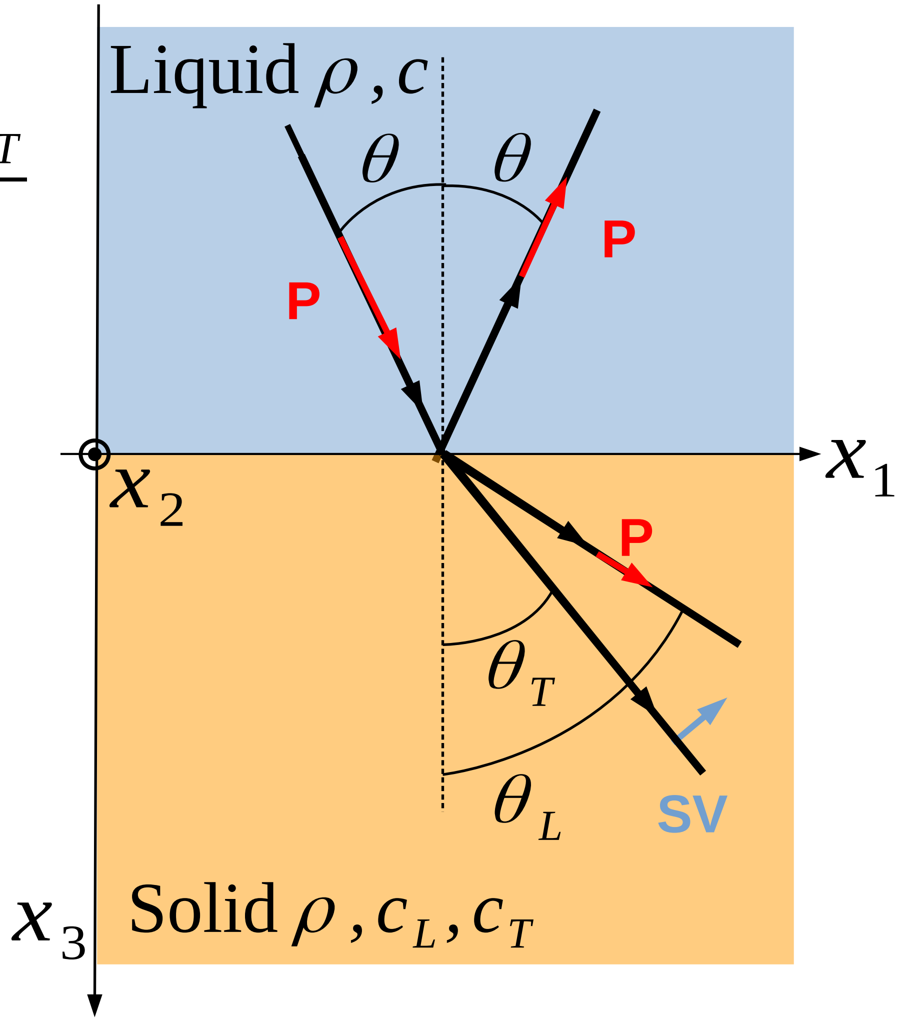
Then reflected compressive waves becomes evanescent.



Solid-Liquid, Liquid-Solid interfaces



$$\frac{\sin\theta}{c} = \frac{\sin\theta_L}{c_L} = \frac{\sin\theta_T}{c_T}$$



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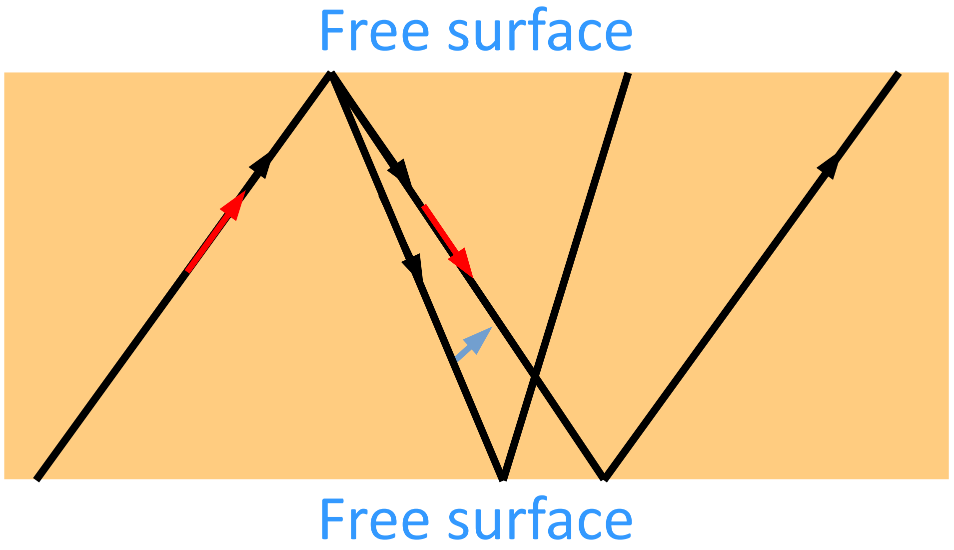
Guided Waves

Specific shapes are waveguides, based on the coupling between propagation properties and boundary conditions

Two free surfaces guide the wave through successive reflections

One free surface is already a waveguide !

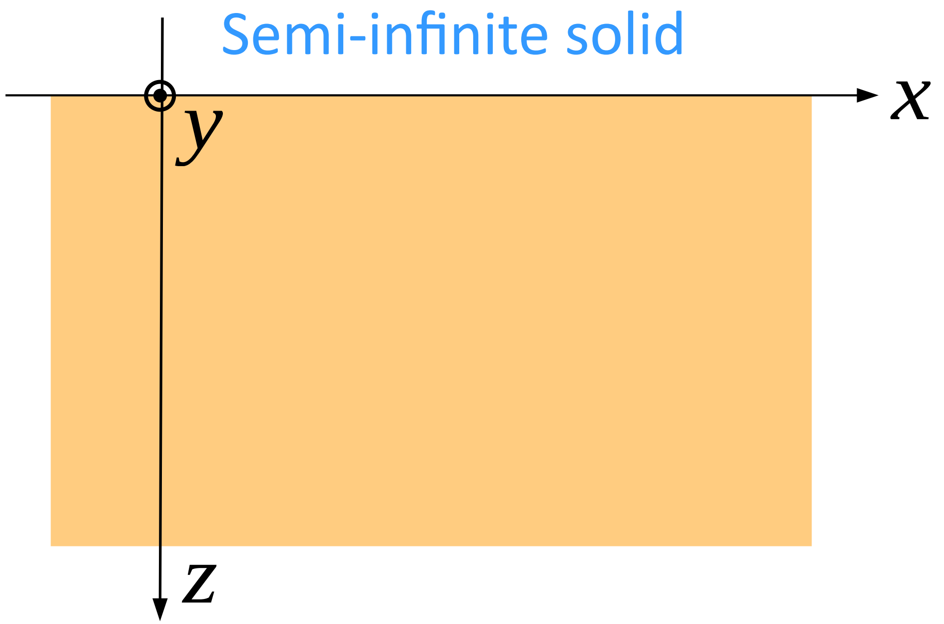


Rayleigh Waves

$$\rho \frac{\partial^2 \vec{u}}{\partial t^2} = (\lambda + \mu)\,\overrightarrow{\text{grad}}\,(\text{div}\,\vec{u}) + \mu\,\vec{\Delta}\,\vec{u}$$

$$\vec{u} = \overrightarrow{\text{grad}}\,\phi_L + \overrightarrow{\text{rot}}\,\vec{\psi}$$

$$\begin{cases} u_x = \dfrac{\partial \phi_L}{\partial x} + \dfrac{\partial \psi_z}{\partial y} - \dfrac{\partial \psi_y}{\partial z} = \dfrac{\partial \phi_L}{\partial x} - \dfrac{\partial \psi_y}{\partial z} \\ u_y = \dfrac{\partial \phi_L}{\partial y} + \dfrac{\partial \psi_x}{\partial z} - \dfrac{\partial \psi_z}{\partial x} = \dfrac{\partial \psi_x}{\partial z} - \dfrac{\partial \psi_z}{\partial x} \\ u_z = \dfrac{\partial \phi_L}{\partial z} + \dfrac{\partial \psi_y}{\partial x} - \dfrac{\partial \psi_x}{\partial y} = \dfrac{\partial \phi_L}{\partial z} + \dfrac{\partial \psi_y}{\partial x} \end{cases}$$



Polarization in (x, z) plane $\Rightarrow u_y = 0$

We can choose

$$\vec{\Psi} = \phi_T\,\vec{e}_y$$

Rayleigh Waves

$$\Delta \phi - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = 0 \quad \rightarrow \quad \frac{\partial^2 \phi_M}{\partial x^2} + \frac{\partial^2 \phi_M}{\partial z^2} - \frac{1}{c_M^2}\frac{\partial^2 \phi_M}{\partial t^2} = 0 \quad \text{with} \quad L = M, T$$

We look for solutions in the form:

Propagative harmonic waves toward x.

$$\phi_M(x, z, t) = f_M(z)\, e^{i(kx - \omega t)}$$

$$\frac{\partial^2 f_M}{\partial z^2} + \left(\frac{\omega^2}{c_M^2} - k^2\right) f_M(z) = 0 \quad \text{with} \quad L = M, T$$

Rayleigh Waves

$$f_M(z) = A_M e^{-\alpha_M z} + B_M e^{\alpha_M z} \quad \text{with} \quad \alpha_M = \sqrt{k^2 - \frac{\omega^2}{c_M^2}} = k\sqrt{1 - \frac{c^2}{c_M^2}}$$

B_M must be zero to avoid divergence.

We can write: $\phi_M(x,z,t) = A_M e^{-\alpha_M z} e^{i(kx-\omega t)}$

$$\vec{u}(x,z,t) = \left[\begin{pmatrix} ik \\ 0 \\ -\alpha_L \end{pmatrix} A_L e^{-\alpha_L z} + \begin{pmatrix} \alpha_T \\ 0 \\ ik \end{pmatrix} A_T e^{\alpha_T z} \right] e^{i(kx-\omega t)}$$

Rayleigh Waves

Boundary conditions: $\underline{\sigma}(\vec{x},t).\vec{e}_z=\vec{0},$

Elasticity law gives:

$$\begin{cases} \sigma_{xz}=\mu\left(\dfrac{\partial u_z}{\partial x}+\dfrac{\partial u_x}{\partial z}\right) \\ \sigma_{yz}=\mu\left(\dfrac{\partial u_z}{\partial y}+\dfrac{\partial u_y}{\partial z}\right) \\ \sigma_{zz}=\lambda\dfrac{\partial u_x}{\partial x}+(\lambda+2\mu)\dfrac{\partial u_z}{\partial z} \end{cases}$$

y invariance and no displacement in y direction cancel this term

Rayleigh Waves

Injecting the waveform:

$$\begin{cases} \sigma_{xx}(x,z,t) = \mu\left[-2ik\alpha_L A_L e^{-\alpha_L z} - (k^2+\alpha_T^2) A_T e^{-\alpha_L z}\right] e^{i(kx-\omega t)} \\ \sigma_{zz}(x,z,t) = \mu\left[(k^2+\alpha_T^2)\alpha_L A_L e^{-\alpha_L z} - 2ik A_T \alpha_T e^{-\alpha_L z}\right] e^{i(kx-\omega t)} \end{cases}$$

At z=0,

$$\begin{cases} 2ik\alpha_L A_L + (k^2+\alpha_T^2) A_T = 0 \\ (k^2+\alpha_T^2)\alpha_L A_L - 2ik\alpha_T A_T = 0 \end{cases}$$

Solving the system provides the dispersion equation:

$$(k^2+\alpha_T^2)^2 - 4k^2\alpha_L\alpha_T = 0$$

Rayleigh Waves

In terms of wave velocity:

$$\left(2-\frac{c^2}{c_T^2}\right)^2-4\sqrt{\left(1-\frac{c^2}{c_L^2}\right)\left(1-\frac{c^2}{c_T^2}\right)}=0$$

Only one solution c_R, depending on c_L and c_T, independent of the frequency

Approximate solution:

$$\frac{c_R}{c_T}=\sqrt{\frac{1.44\,\lambda+0.88\,\mu}{1.58\,\lambda+1.16\,\mu}}$$

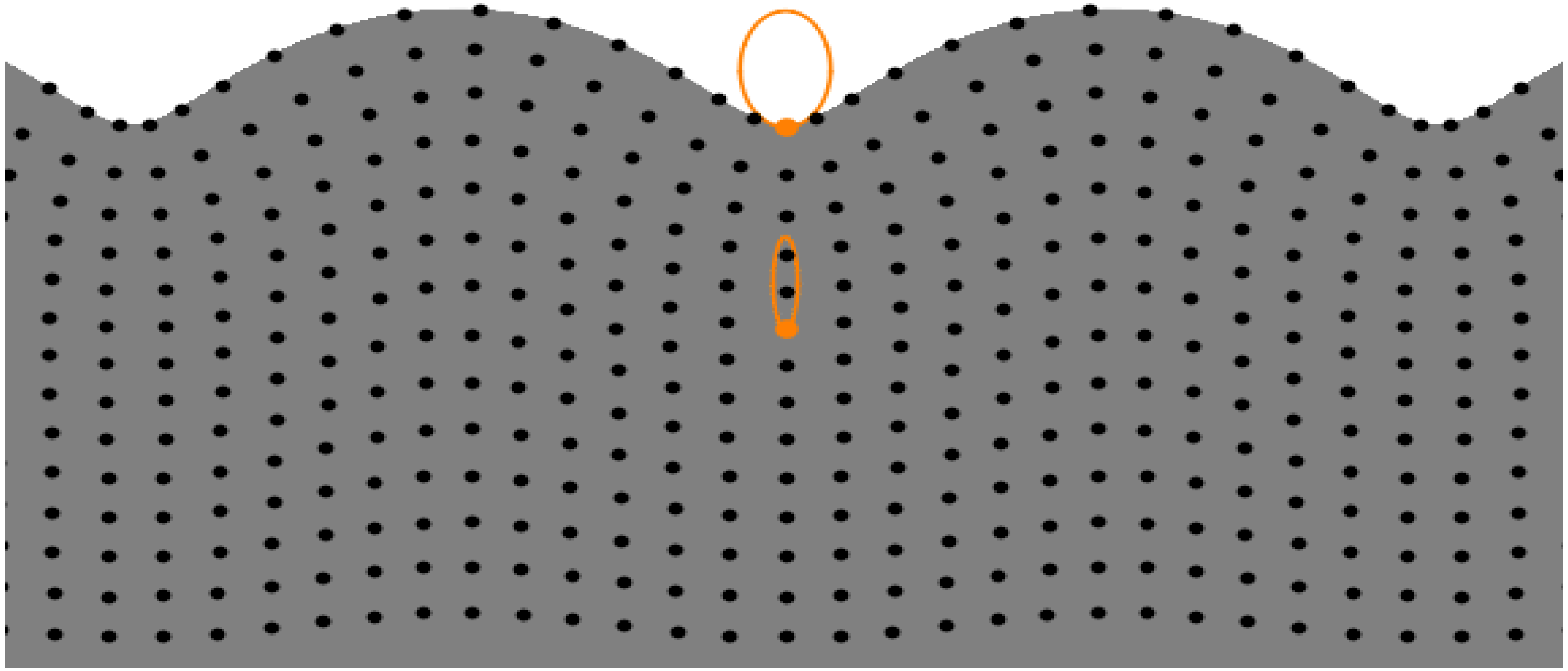
Rayleigh Waves

$$\begin{cases} U_x(x,z,t) = -\left(k e^{-\alpha_L z} - \sqrt{\alpha_L \alpha_T}\, e^{-\alpha_T z}\right) A_L \sin(kx - \omega t) \\ U_z(x,z,t) = \left(-\alpha_L e^{-\alpha_L z} + k\sqrt{\dfrac{\alpha_L}{\alpha_T}}\, e^{-\alpha_T z}\right) A_L \cos(kx - \omega t) \end{cases}$$

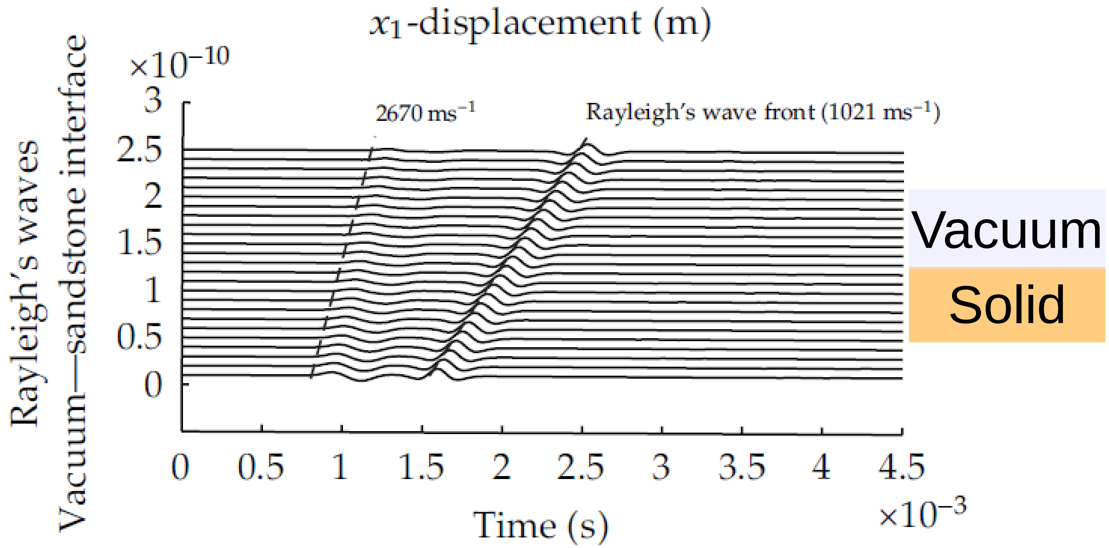
- Evanescent part in z direction
- Phase quadrature between x and z: elliptic polarization
- Possible sign changes

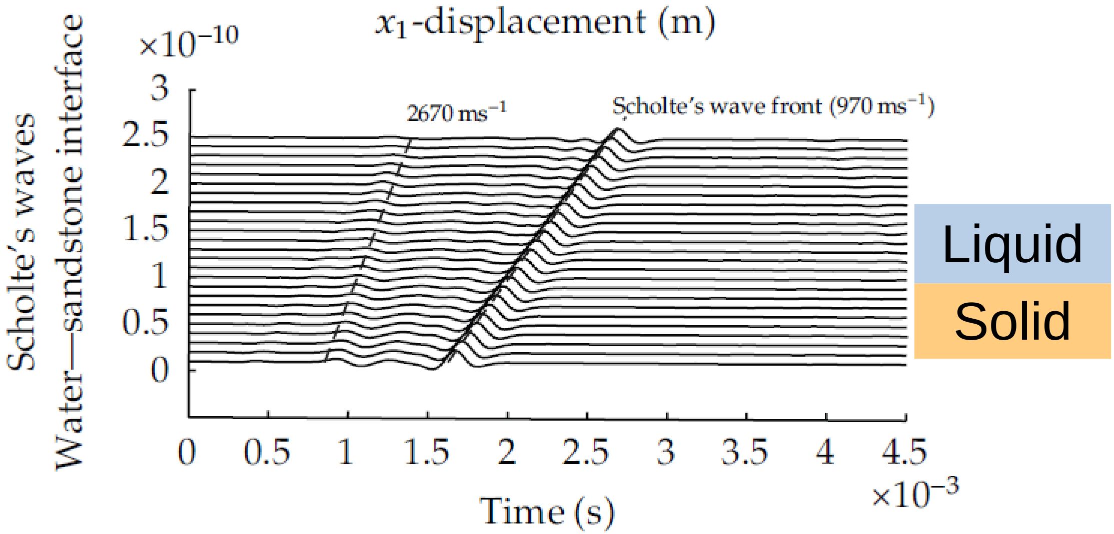
Rayleigh Waves

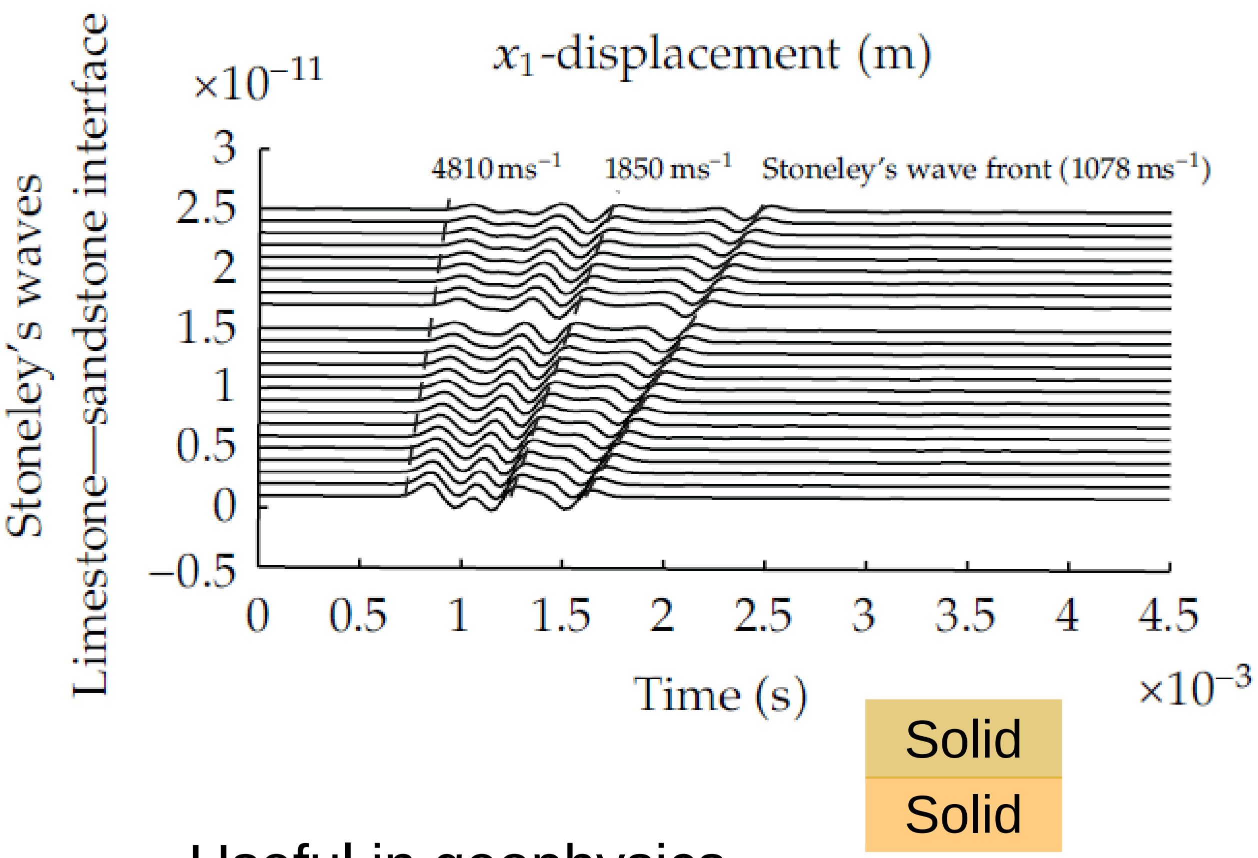
$$\begin{cases} U_x(x,z,t) = -\left(k e^{-\alpha_L z} - \sqrt{\alpha_L \alpha_T}\, e^{-\alpha_T z}\right) A_L \sin(kx - \omega t) \\ U_z(x,z,t) = \left(-\alpha_L e^{-\alpha_L z} + k\sqrt{\frac{\alpha_L}{\alpha_T}}\, e^{-\alpha_T z}\right) A_L \cos(kx - \omega t) \end{cases}$$



Scholte and Stoneley Waves





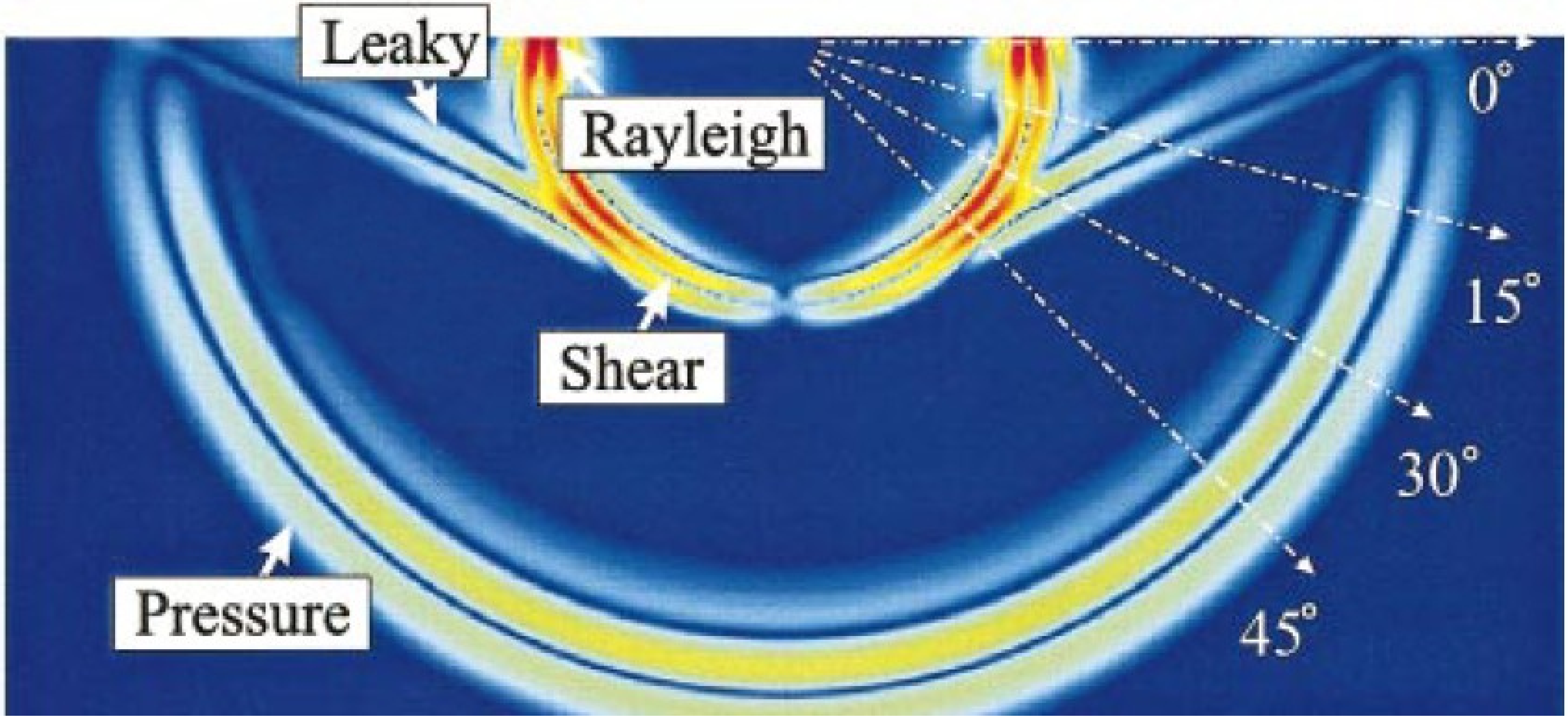


Useful in geophysics

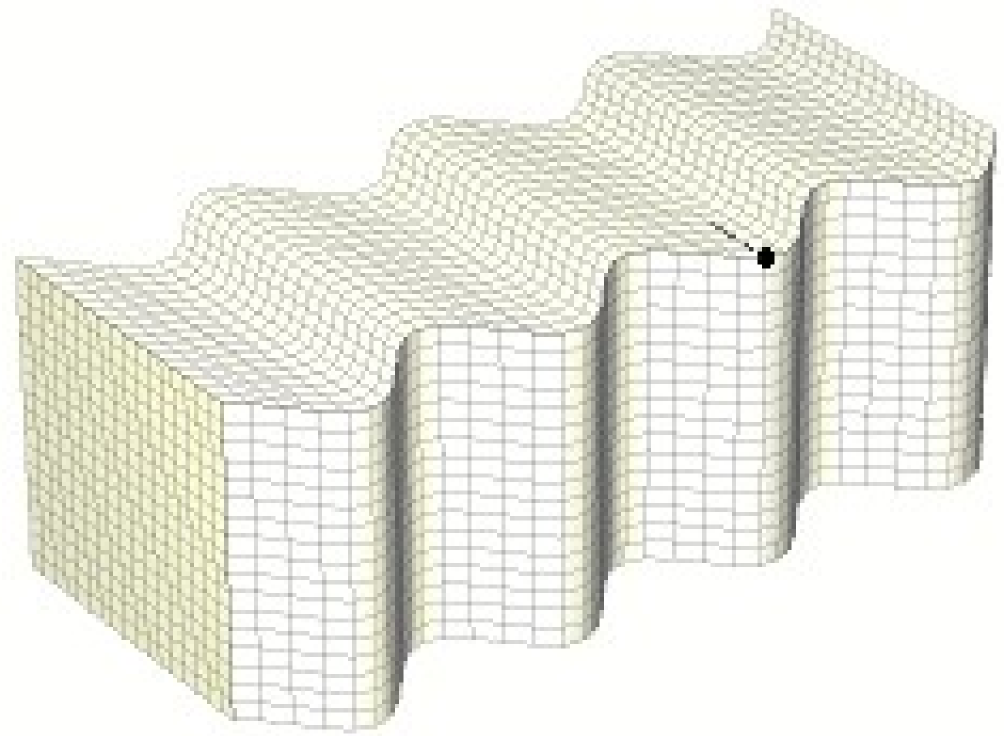
Flores-Mendez et al. Journ. Appl. Math 2011

Leaky Waves

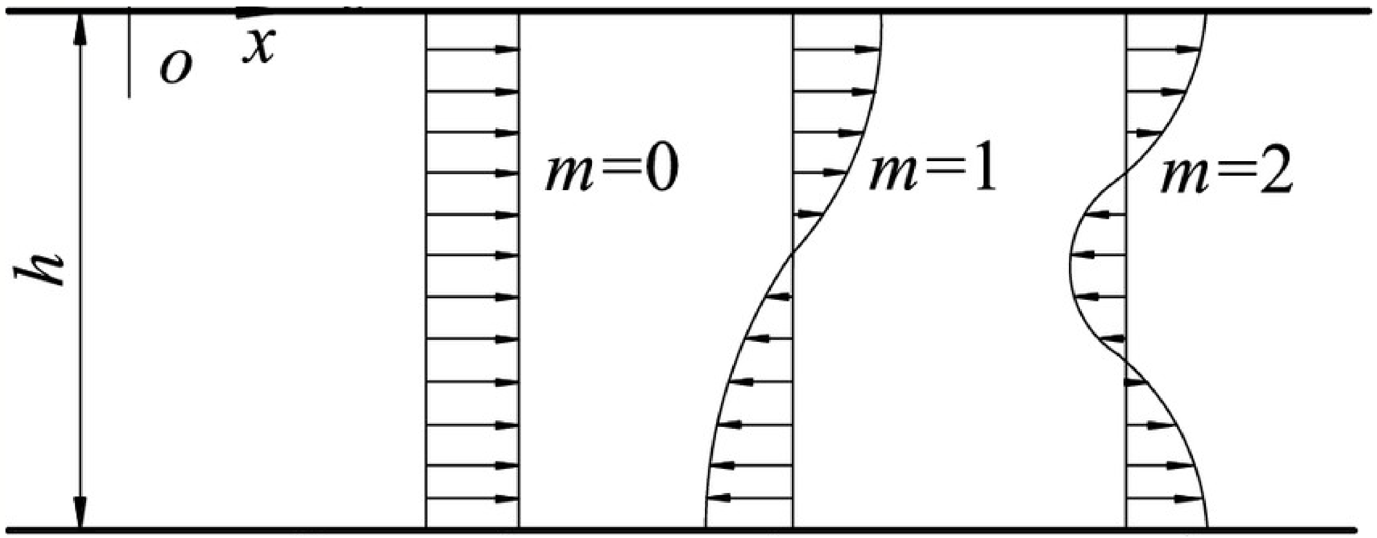
Surface guided wave, decreasing with z and x



SH Guided Waves

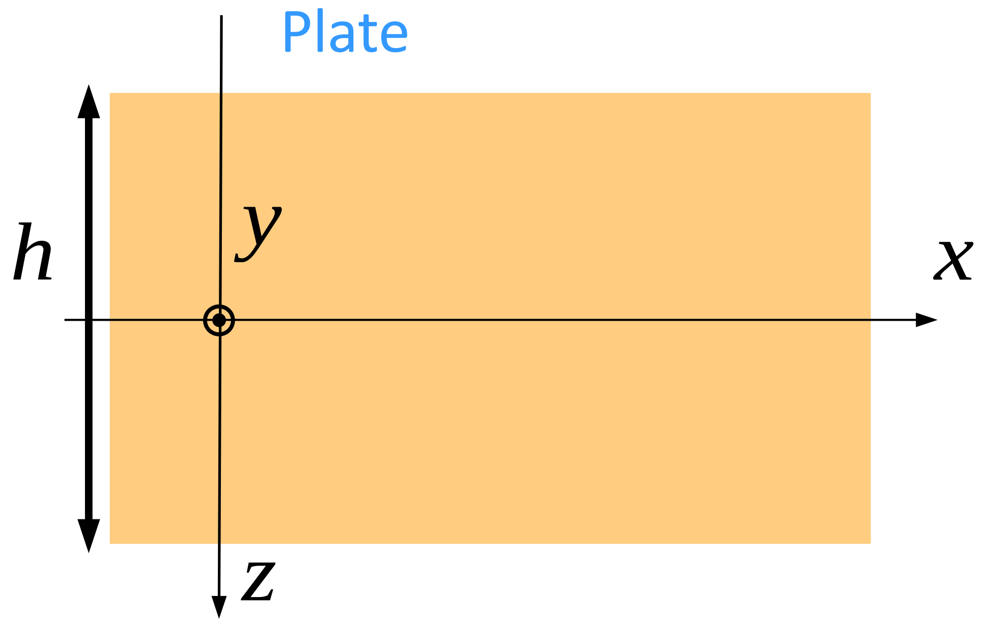


From Noé Jiménez website
https://nojigon.webs.upv.es/



Cut-off at low frequency for m > 0

Lamb Waves



Polarization in (x, z) plane $\Rightarrow u_y = 0$

We use $\vec{\Psi} = \phi_T \vec{e}_y$

$$\begin{cases} u_x = \dfrac{\partial \phi_L}{\partial x} - \dfrac{\partial \phi_T}{\partial z} \\ u_z = \dfrac{\partial \phi_L}{\partial z} + \dfrac{\partial \phi_T}{\partial x} \end{cases}$$

Looking for propagative form in x:

$$\begin{cases} \phi_L(x,z,t) = f_L(z) e^{i(kx-\omega t)} \\ \phi_T(x,z,t) = f_T(z) e^{i(kx-\omega t)} \end{cases},$$

then

$$\begin{cases} \dfrac{\partial^2 \phi_L}{\partial z^2} + p^2 \phi_L = 0 \\ \dfrac{\partial^2 \phi_T}{\partial z^2} + q^2 \phi_T = 0 \end{cases} \qquad \begin{cases} p^2 = \dfrac{\omega^2}{c_L^2} - k^2 = k_L^2 - k^2 \\ q^2 = \dfrac{\omega^2}{c_T^2} - k^2 = k_T^2 - k^2 \end{cases}$$

Lamb Waves

Deriving the displacement from the potentials:

$$\begin{cases} u_x = \dfrac{\partial \phi_L}{\partial x} - \dfrac{\partial \phi_T}{\partial z} = i k \phi_L - \dfrac{\partial \phi_T}{\partial z} \\ u_z = \dfrac{\partial \phi_L}{\partial z} + \dfrac{\partial \phi_T}{\partial x} = \dfrac{\partial \phi_L}{\partial z} + i k \phi_T \end{cases}$$

We only need these two components of the stress to express boundary conditions:

$$\begin{cases} \sigma_{xz} = \mu \left(\dfrac{\partial u_z}{\partial x} + \dfrac{\partial u_x}{\partial z} \right) = \mu \left((q^2 - k^2) \phi_T + 2 i k \dfrac{\partial \phi_L}{\partial z} \right) \\ \sigma_{zz} = \lambda \dfrac{\partial u_x}{\partial x} - (\lambda + 2\mu) \dfrac{\partial u_z}{\partial z} = \mu \left((k^2 - q^2) \phi_L + 2 i k \dfrac{\partial \phi_T}{\partial z} \right) \end{cases}$$

Lamb Waves

Boundary conditions:

$$\begin{cases} \sigma_{xz}(z=\pm h/2)=0 \\ \sigma_{zz}(z=\pm h/2)=0 \end{cases}$$

Can be verified only if the two potentials have different parities:

$$f_L(z)=B\cos(pz+\alpha) \quad \text{and} \quad f_T(z)=A\sin(qz+\alpha) \quad \text{with} \quad \alpha=0 \quad \text{or} \quad \pi/2$$

Symmetrical modes — Anti-symmetrical modes

$$\begin{cases} u_x=\dfrac{\partial \phi_L}{\partial x}-\dfrac{\partial \phi_T}{\partial z}=\left(ikB\cos(pz+\alpha)-qA\cos(qz+\alpha)\right)e^{i(kx-\omega t)} \\ u_z=\dfrac{\partial \phi_L}{\partial z}+\dfrac{\partial \phi_T}{\partial x}=\left(pB\sin(pz+\alpha)+ikA\sin(qz+\alpha)\right)e^{i(kx-\omega t)} \end{cases}$$

Lamb Waves

Boundary conditions:

$$\begin{cases} \sigma_{xz}(z=\pm h/2)=0 \\ \sigma_{zz}(z=\pm h/2)=0 \end{cases}$$

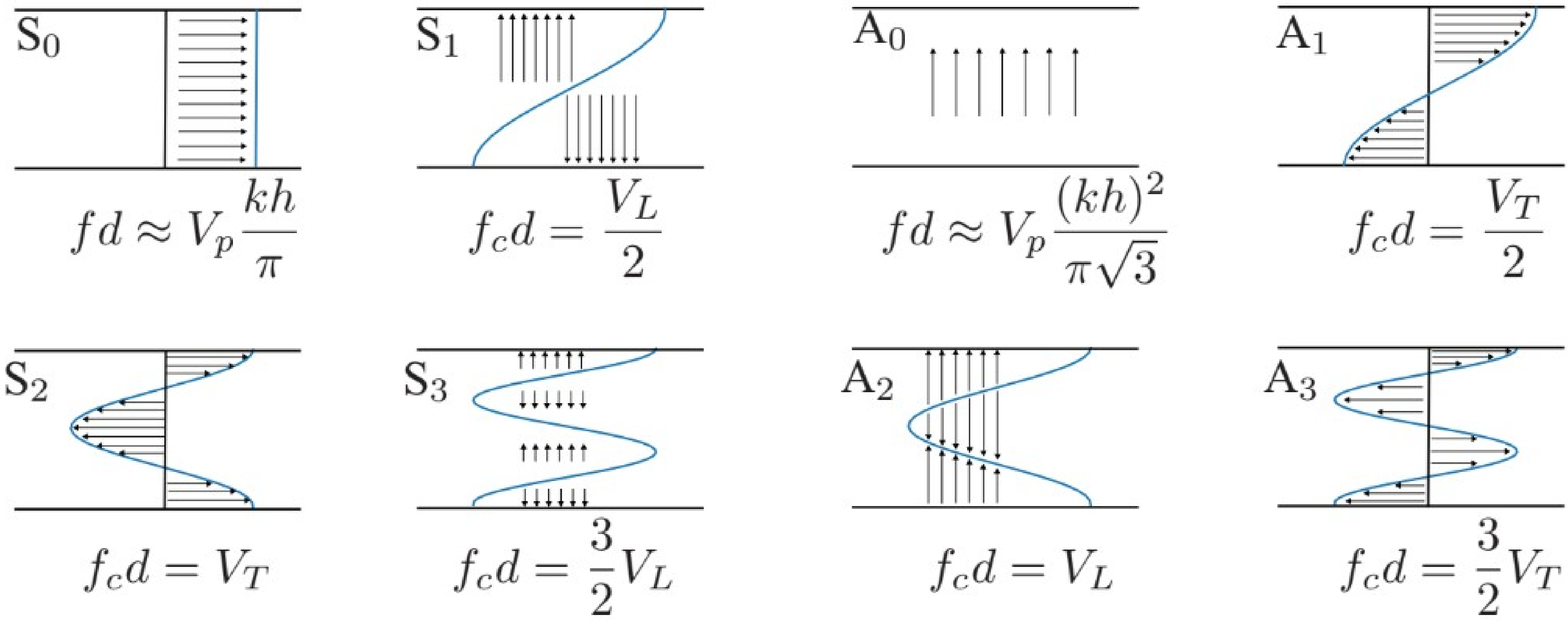
$$\begin{cases} \sigma_{xz}=\mu\left((q^2-k^2)\phi_T+2ik\dfrac{\partial \phi_L}{\partial z}\right) \\ \sigma_{zz}=\mu\left((k^2-q^2)\phi_L+2ik\dfrac{\partial \phi_T}{\partial z}\right) \end{cases}$$

$$\begin{cases} (q^2-k^2)A\sin(qh/2+\alpha)-2ikBp\sin(ph/2+\alpha)=0 \\ (k^2-q^2)B\cos(ph/2+\alpha)+2ikAq\cos(qh/2+\alpha)=0 \end{cases}$$

Solving the system gives the dispersion relation of both modes:

$$\frac{\omega^4}{c_T^4}=4k^2q^2\left(1-\frac{p}{q}\frac{\tan(ph/2+\alpha)}{\tan(qh/2+\alpha)}\right) \quad \text{with} \quad \alpha=0 \quad \text{or} \quad \pi/2$$

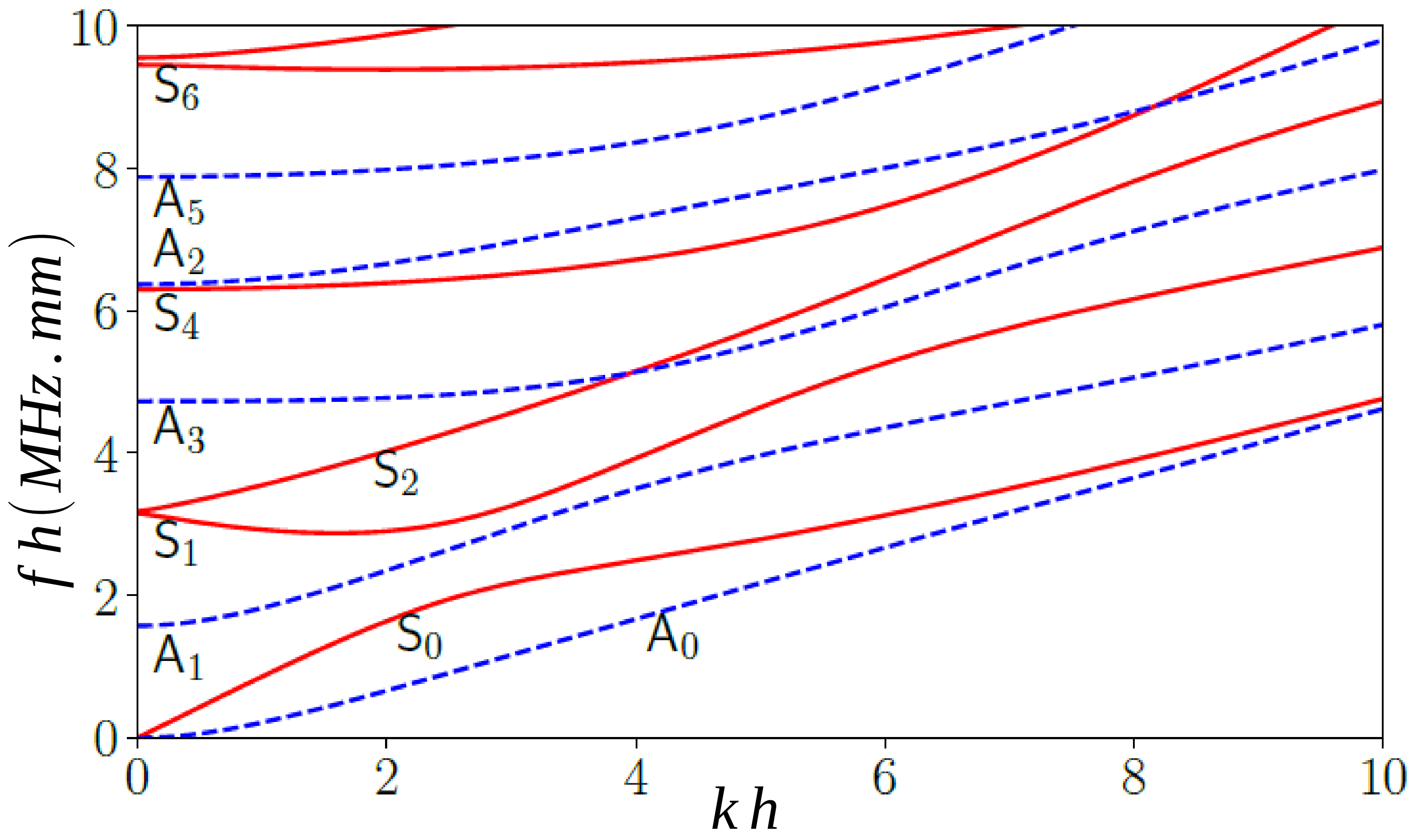
Lamb Waves



Symmetrical modes

Anti-symmetrical modes

Lamb Waves



- No crossings for each type of mode
- Cut-off frequency for all but the first modes A_0 and S_0
- Growing with k most of the time

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