Introduction to Acoustics

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Content

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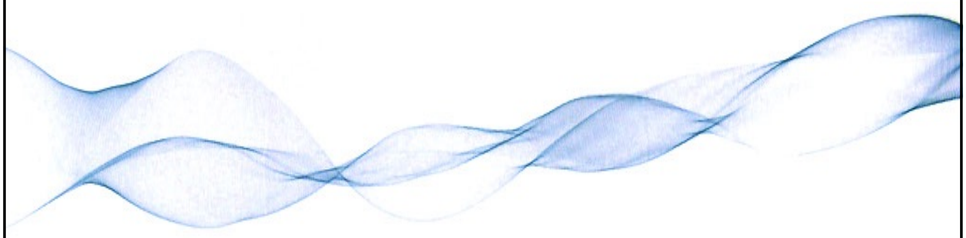
Sound waves in fluids



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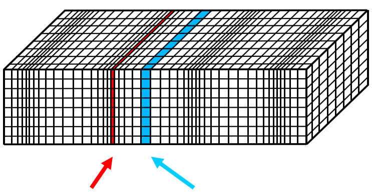
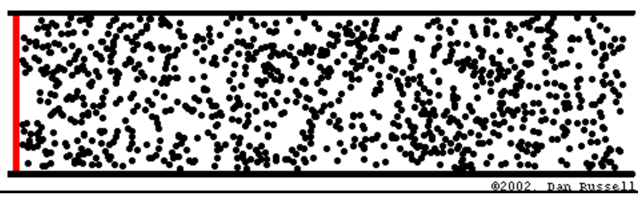
Sound waves

The sound wave in a fluid is associated with compression and dilatation of the fluid. Those compression and dilatation are associated with variation in pressure (and in temperature).





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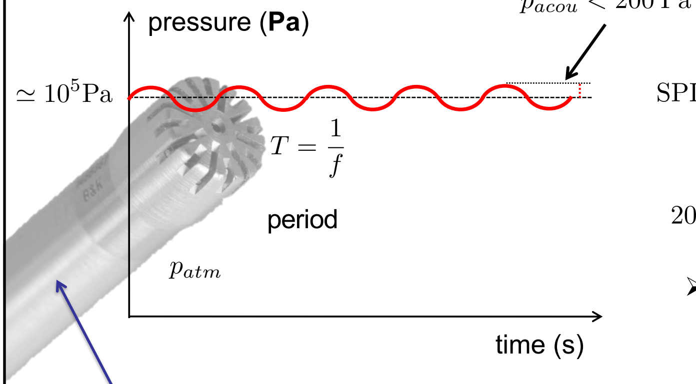
Sound waves



$$\mathrm{SPL} = 20 \log_{10}\left(\frac{p}{p_{ref}}\right)$$

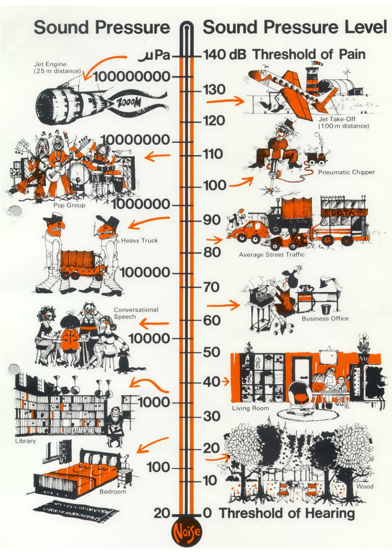
$$p_{ref} = 2.10^{-5}\mathrm{Pa}$$

$$200\ \mathrm{Pa} \mapsto 140\ \mathrm{dB\ SPL}$$

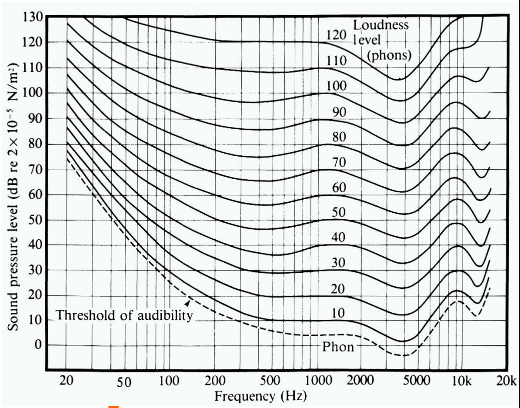
- Threshold of Pain ~ 130 dB SPL

The pressure is measured by microphones

Sound waves



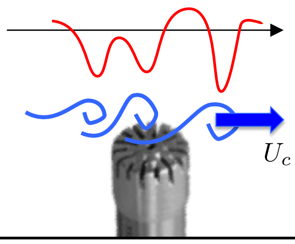
Equal-loudness contours for the human ear,

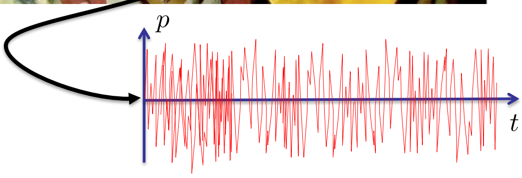


Links with psychoacoustics

Sound waves

All pressure fluctuations are not sound





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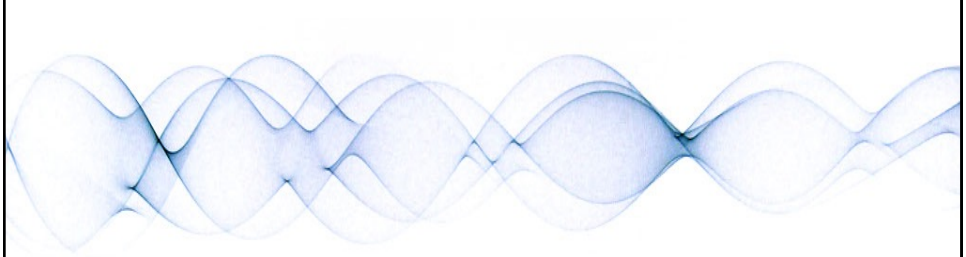
Waves Motion



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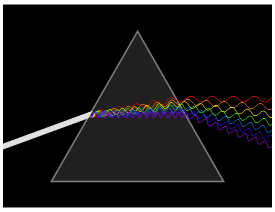
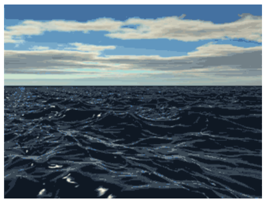
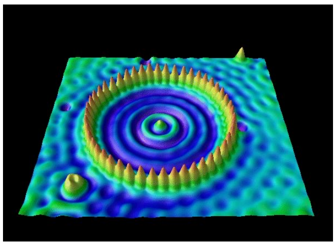
Waves motion



Light, electomagnetic waves



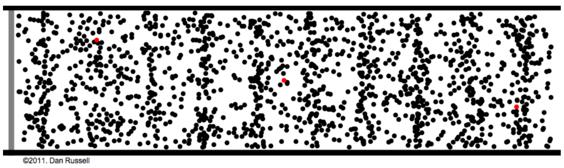
Water waves



Quantum waves

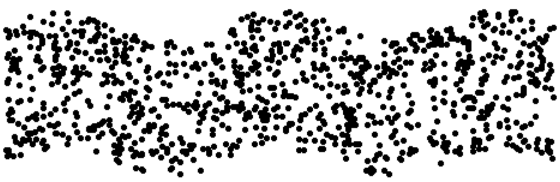
“La Ola” or “Mexican wave”

Waves motion

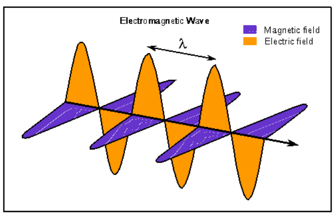


Longitudinal Wave

Propagation

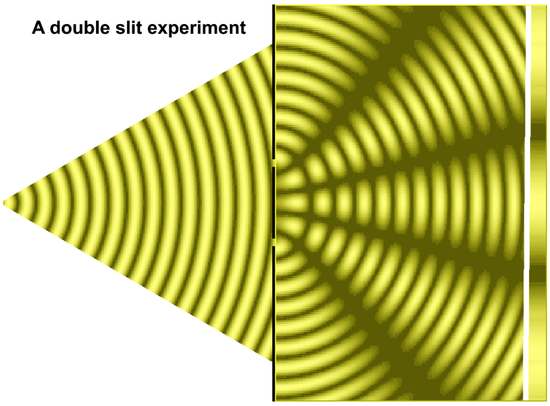


Transverse Wave



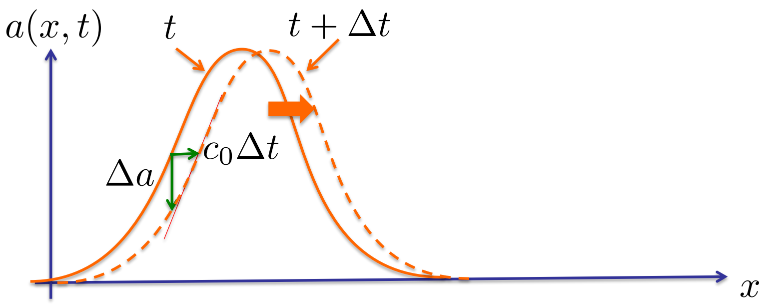
Waves motion

A double slit experiment



Diffraction,
Interferences, ..

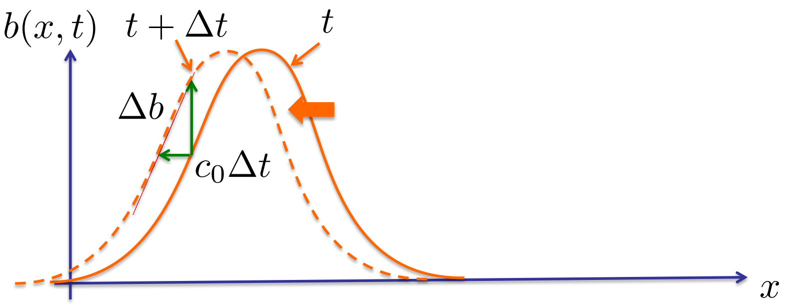
Waves motion



$$\frac{\partial a}{\partial x} = -\frac{\partial a}{\partial (c_0 t)} = -\frac{1}{c_0}\frac{\partial a}{\partial t}$$

$$\rightarrow a(x,t) = a(x - c_0 t)$$

Waves motion



$$\frac{\partial b}{\partial x} = \frac{\partial b}{\partial (c_0 t)} = \frac{1}{c_0}\frac{\partial b}{\partial t}$$

$$\rightarrow \quad b(x,t) = b(x + c_0 t)$$

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Waves motion

$$\frac{\partial^2 a}{\partial x^2} = \frac{\partial}{\partial x}\left(-\frac{\partial a}{\partial (c_0 t)}\right) = \frac{1}{c_0^2}\frac{\partial^2 a}{\partial t^2}$$

$$\frac{\partial^2 b}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial b}{\partial (c_0 t)}\right) = \frac{1}{c_0^2}\frac{\partial^2 b}{\partial t^2}$$

1-D Wave equation $\rightarrow$ $$\frac{\partial^2 p}{\partial x^2} = \frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2}$$

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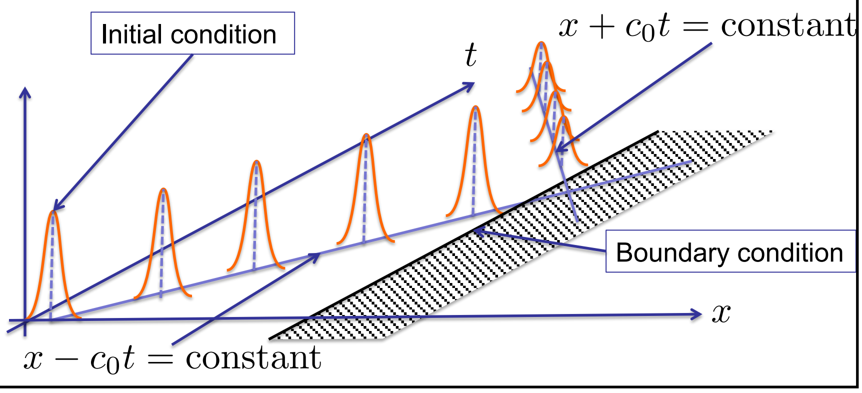
Waves motion

1-D Wave equation

$$\frac{\partial^2 p}{\partial x^2} - \frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} = 0 \Rightarrow p(x,t) = a(x - c_0 t) + b(x + c_0 t)$$



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General equations

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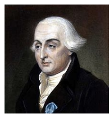
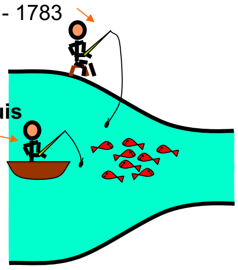
General equations

Leonhard Euler
1707 - 1783



Joseph-Louis Lagrange
1736-1813



- mass conservation

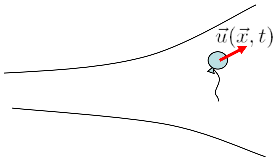
All variable are in Eulerian description:

$\vec{u} = \vec{u}(\vec{x}, t)$ fixed point

The other possibility is the Lagrangian description where a particle is following in its motion (leads to Galbrun equation).

$$\frac{1}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t} + \vec{\nabla}.\vec{u} = 0 \text{ where } \frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \vec{u}.\vec{\nabla}$$

convective (or total) derivative



$$\frac{\partial \rho}{\partial t} + \vec{\nabla}.(\rho\vec{u}) = 0$$

General equations

- Momentum conservation law

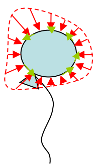
$$\rho\frac{\mathrm{D}\vec{u}}{\mathrm{D}t} = -\vec{\nabla}p + \vec{\nabla}\vec{\vec{\tau}} + \vec{f}$$

Mass

Acceleration

Surface forces

Volume forces



Viscous stress $\vec{\vec{\tau}} = \eta\left(\vec{\vec{\nabla}}\vec{u} + (\vec{\vec{\nabla}}\vec{u})^T\right) - \frac{2}{3}\eta\left(\vec{\nabla}.\vec{u}\right)\vec{\vec{I}}$

Dynamic viscosity

When viscous effects are neglected, leads to **Euler equation**:

$$\rho\frac{\mathrm{D}\vec{u}}{\mathrm{D}t} = -\vec{\nabla}p$$

Leonhard Euler (1707-1783)

General equations

- Energy conservation law

$$\rho T \frac{\mathrm{D}s}{\mathrm{D}t} = -\vec{\nabla}\vec{q} - \vec{\vec{\tau}} : \vec{u}$$

entropy   heat flux $\vec{q} = \kappa \vec{\nabla} T$

- Constitutive equation

$$\frac{\mathrm{D}p}{\mathrm{D}t} = \left(\frac{\partial p}{\partial \rho}\right)_s \frac{\mathrm{D}\rho}{\mathrm{D}t} + \left(\frac{\partial p}{\partial s}\right)_\rho \frac{\mathrm{D}s}{\mathrm{D}t}$$

Where $c_0^2 = \left(\frac{\partial p}{\partial \rho}\right)_s$ c_0 is the isentropic **speed of sound**

General equations

- For a perfect fluid :   No viscous effects
  No heat flux
  ⇒ Isentropic flow

Mass conservation $\frac{\mathrm{D}\rho}{\mathrm{D}t} = -\rho \vec{\nabla}.\vec{u}$

Euler's equation $\rho \frac{\mathrm{D}\vec{u}}{\mathrm{D}t} = -\vec{\nabla} p$

Isentropic flow $\frac{\mathrm{D}\rho}{\mathrm{D}t} = \frac{1}{c_0^2} \frac{\mathrm{D}p}{\mathrm{D}t}$

Unknowns: $\vec{u}$, p, ρ

$$\begin{cases} \frac{1}{c_0^2} \frac{\mathrm{D}p}{\mathrm{D}t} = -\rho \vec{\nabla}.\vec{u} \\ \rho \frac{\mathrm{D}\vec{u}}{\mathrm{D}t} = -\vec{\nabla} p \end{cases}$$

Set of non linear equations

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General equations

- Linearization

In air at normal condition p_0 = 1.10[5] Pa and ρ_0 = 1.3 kg/m[3]
for f = 1 kHz, the acoustical perturbation is of the order:

SPL	p' (Pa)	v' (m/s)	δ' (m)	ρ' (kg/m³)
140 dB	200	0.5	8.10⁻⁵	2.10⁻³
0 dB	2.10⁻⁵	5.10⁻⁸	1.10⁻¹¹	2.10⁻¹⁰

$\vec{u} = \vec{u}_0 + \varepsilon \vec{u}'$ $\quad p = p_0 + \varepsilon p'$ $\quad \rho = \rho_0 + \varepsilon \rho'$ $\quad$ where $\quad \varepsilon << 1$

Mean flow:
Time independent
(more complex when turbulence is taken into account)

Perturbations:
Perturbations are not necessarily “acoustics”.
It could also be “hydrodynamics” or “entropy” waves

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General equations

- quiescent ($u_0 = 0$) inviscid and non-conducting fluid

$$\begin{cases} \dfrac{\partial \rho'}{\partial t} = -\rho_0 \vec{\nabla}.\vec{u}' \\ \rho_0 \dfrac{\partial \vec{u}'}{\partial t} = -\vec{\nabla} p' \\ p' = c_0^2 \rho' \end{cases}$$

$$\begin{cases} \dfrac{\partial}{\partial t} \left[\dfrac{1}{c_0^2} \dfrac{\partial p'}{\partial t} = -\rho_0 \vec{\nabla}.\vec{u}' \right] \\ \vec{\nabla} \left[\rho_0 \dfrac{\partial \vec{u}'}{\partial t} = -\vec{\nabla} p' \right] \end{cases}$$

$$\frac{1}{c_0^2} \frac{\partial^2 p'}{\partial t^2} - \Delta p' = 0$$

Wave equation

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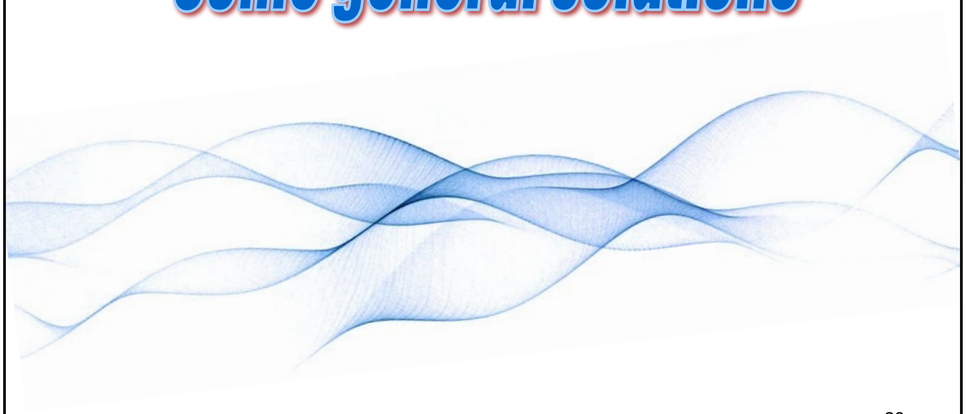
Some general solutions



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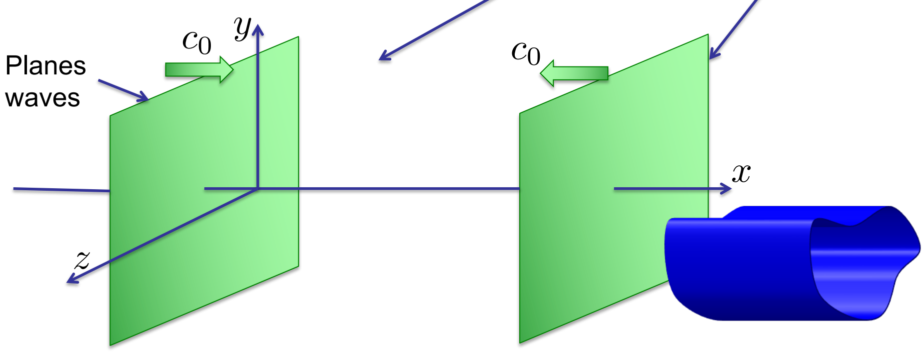
Some general solutions

- Equation in 1D $\dfrac{1}{c_0^2}\dfrac{\partial^2 p'}{\partial t^2} - \Delta p' = 0$ + propagation along x

$$\frac{\partial^2 p}{\partial x^2} - \frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} = 0 \Rightarrow p(x,t) = p^+(t - \frac{x}{c_0}) + p^-(t + \frac{x}{c_0})$$



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Some general solutions

- Equation in 1D
  Velocity $\vec{u} = u(x,t)\vec{u}$

$$\frac{1}{c_0^2}\frac{\partial p}{\partial t} = -\rho_0 \frac{\partial u}{\partial x}$$

$$p(x,t) = p^+(t - \frac{x}{c_0}) = p^+(T)$$

where $T = t - \frac{x}{c_0}$

Emission time

$$u(x,t) = u^+(t - \frac{x}{c_0}) = u^+(T)$$

$$\frac{\partial p}{\partial t} = \frac{\mathrm{d}p^+}{\mathrm{d}T}\frac{\partial T}{\partial t} = \frac{\mathrm{d}p^+}{\mathrm{d}T}$$

$$\frac{\partial u}{\partial x} = \frac{\mathrm{d}u^+}{\mathrm{d}T}\frac{\partial T}{\partial x} = -\frac{1}{c_0}\frac{\mathrm{d}u^+}{\mathrm{d}T}$$

$$\frac{\mathrm{d}u^+}{\mathrm{d}T} = \frac{1}{\rho_0 c_0}\frac{\mathrm{d}p^+}{\mathrm{d}T}$$

$$u^+ = \frac{p^+}{\rho_0 c_0}$$

Some general solutions

- Equation in 1D

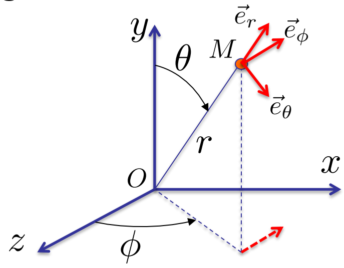
$$u^+ = \frac{p^+}{\rho_0 c_0}$$

On the same way $u^- = -\frac{p^-}{\rho_0 c_0}$

$$p = p^+ + p^- \rightarrow u = \frac{1}{\rho_0 c_0}(p^+ - p^-)$$

Some general solutions

- Spherical waves



$$\vec{F} = F_r\vec{e}_r + F_\theta\vec{e}_\theta + F_\phi\vec{e}_\phi$$

General expression of the operators in spherical coordinates

$$\vec{\nabla} f = \frac{\partial f}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\vec{e}_\theta + \frac{1}{r sin(\theta)}\frac{\partial f}{\partial \phi}\vec{e}_\phi$$

$$\vec{\nabla}.\vec{F} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 F_r\right) + \frac{1}{r sin(\theta)}\frac{\partial}{\partial \theta}\left(\sin(\theta)F_\theta\right) + \frac{1}{r sin(\theta)}\frac{\partial F_\phi}{\partial \phi}$$

$$\Delta f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2 sin(\theta)}\frac{\partial}{\partial \theta}\left(sin(\theta)\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2 sin(\theta)}\frac{\partial^2 f}{\partial \phi^2}$$

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Some general solutions

- Spherical waves

If the pressure only depends on the distance r to the centre O

$$\frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} - \Delta p = 0 \Rightarrow \frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} - \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial p}{\partial r}\right) = 0$$

using $$\frac{1}{r}\frac{\partial}{\partial r}\left(r^2\frac{\partial p}{\partial r}\right) = 2\frac{\partial p}{\partial r} + r\frac{\partial^2 p}{\partial r^2} = \frac{\partial^2 (rp)}{\partial r^2}$$

leads to $$\boxed{\frac{1}{c_0^2}\frac{\partial^2 (rp)}{\partial t^2} - \frac{\partial^2 (rp)}{\partial r^2} = 0}$$

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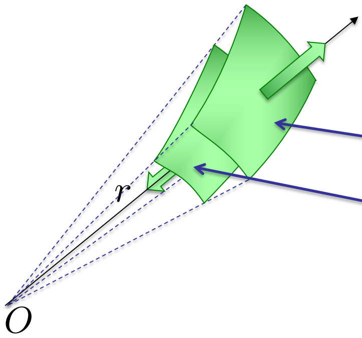
Some general solutions

- Spherical waves

$$\frac{1}{c_0^2}\frac{\partial^2(rp)}{\partial t^2} - \frac{\partial^2(rp)}{\partial r^2} = 0$$

Wave equation for rp



$$rp = a(t - \frac{r}{c_0}) + b(t + \frac{r}{c_0})$$

Outgoing waves

Incoming waves

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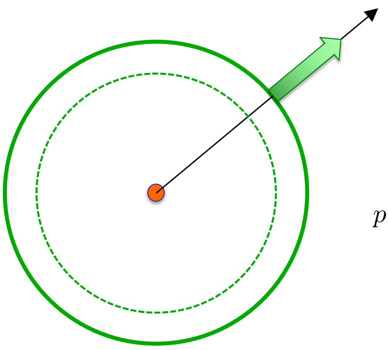
Some general solutions

- Spherical waves in free field

No waves are incoming from infinity → $b(t + \frac{r}{c_0}) = 0$ **Sommerfeld** radiation condition



$$rp = a(t - \frac{r}{c_0}) = a(T)$$

By assuming an harmonic motion :

$$a(T) = A_0 \cos(\omega T)$$

$$p = \frac{A_0}{r}\cos(\omega t - k_0 r) \quad \text{where} \quad k_0 = \frac{\omega}{c_0}$$

Wave number

Some general solutions

- Spherical waves in free field

$$p = \frac{A_0}{r}\cos(\omega t - k_0 r)$$

$$\begin{aligned} u_r &= \vec{u}.\vec{e}_r = |u_r|\cos(\omega t - k_0 r + \phi) \\ &= |u_r|\left(\cos(\omega t - k_0 r)\cos(\phi) - \sin(\omega t - k_0 r)\sin(\phi)\right) \end{aligned}$$

It is easier to use the complex form :

$$p = \mathrm{Real}\left(\frac{A_0}{r}\exp\left(\mathrm{j}(\omega t - k_0 r)\right)\right)$$

$$u_r = \mathrm{Real}\left(\hat{u}_r \exp\left(\mathrm{j}(\omega t - k_0 r)\right)\right)$$

where $\hat{u}_r \in \mathbb{C}$

$$\left(\rho_0 \frac{\partial \vec{u}}{\partial t} - \vec{\nabla} p\right).\vec{e}_r = 0$$

$$-\rho_0 \mathrm{j}\omega \hat{u}_r = \frac{\partial p}{\partial r}$$

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Some general solutions

- Spherical waves in free field

The Euler equation
along the radial direction:

$$\left(\rho_0 \frac{\partial \vec{u}}{\partial t} + \vec{\nabla} p\right).\vec{e}_r = 0$$

$$\rho_0 \frac{\partial u_r}{\partial t} = -\frac{\partial p}{\partial r}$$

Leads to:

$$\mathrm{j}\omega\rho_0\hat{u}_r \exp\left(\mathrm{j}(\omega t - k_0 r)\right) = -\frac{\partial p}{\partial r} = \left(-\mathrm{j}k_0\frac{A_0}{r} + \frac{A_0}{r^2}\right)\exp\left(\mathrm{j}(\omega t - k_0 r)\right)$$

$$\hat{u}_r = \frac{\hat{p}}{\rho_0 c_0}\left(1 + \frac{1}{\mathrm{j}k_0 r}\right)$$

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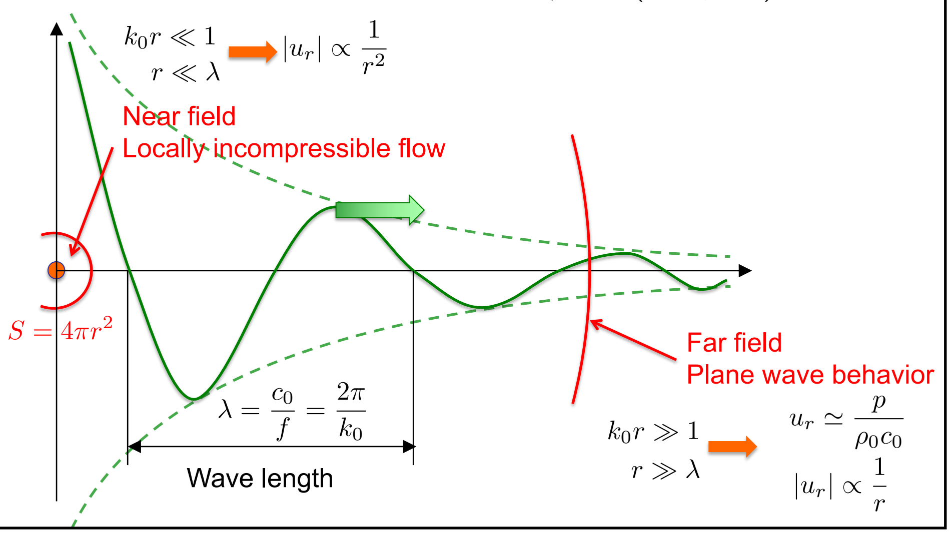
Some general solutions

- Spherical waves in free field $u_r = \frac{p}{\rho_0 c_0}\left(1 + \frac{1}{\mathrm{j}k_0 r}\right)$



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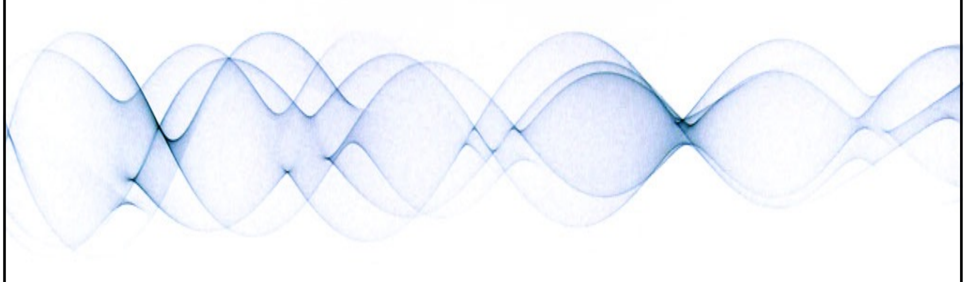
Fourier transforms



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Fourier transforms

- Definition

The Fourier transform is defined by:

$$\hat{p}(\omega) \stackrel{\text{def}}{=} \frac{1}{2\pi} \int_{-\infty}^{\infty} p(t)\, e^{-\mathrm{j}\omega t}\, \mathrm{d}t$$

and the inverse Fourier transform is defined by:

$$p(t) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} \hat{p}(\omega)\, e^{\mathrm{j}\omega t}\, \mathrm{d}\omega$$

Joseph Fourier
(1768 - 1830)

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Fourier transforms

The Fourier transform of $\frac{\mathrm{d}p}{\mathrm{d}t}$ is equal to $\mathrm{j}\omega\, \hat{p}(\omega)$

The wave equation becomes the Helmholtz equation

$$\Delta p - \frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} = 0 \quad \xrightarrow{\text{F.T.}} \quad \Delta \hat{p} + \frac{\omega^2}{c_0^2}\hat{p} = 0$$

$$\Delta \hat{p} + k_0^2\, \hat{p} = 0$$

Helmholtz equation

Hermann von Helmholtz
(1821 -1894)

Whoever in the pursuit of science, seeks after immediate practical utility may rest assured that he seeks in vain. (H. von Helmholtz, 1862)

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Fourier transforms

The spatial Fourier transform is defined by:

$$\check{p}(k,t) \stackrel{\text{def}}{=} \frac{1}{2\pi}\int_{-\infty}^{\infty} p(x,t)\, e^{\mathrm{j}kx}\, \mathrm{d}x$$

and the inverse spatial Fourier transform is defined by:

$$p(x,t) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} \check{p}(k,t)\, e^{-\mathrm{j}kx}\, \mathrm{d}k$$

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Fourier transforms

Example: Calculation on the 1D wave equation with a source concentrated in time and in space

$$\frac{\partial^2 p}{\partial t^2} - c_0^2 \frac{\partial^2 p}{\partial x^2} = \delta(x)\delta(t)$$ where $\delta(x)$, $\delta(t)$ are the Dirac generalized functions

The Fourier Transforms in time and space lead to:

$$-\omega^2 \hat{p} + c_0^2 k^2 \hat{p} = \frac{1}{4\pi^2}$$

$$\Rightarrow \hat{p}(k,\omega) = \frac{1}{4\pi^2 c_0^2}\,\frac{1}{k^2 - \omega^2/c_0^2}$$

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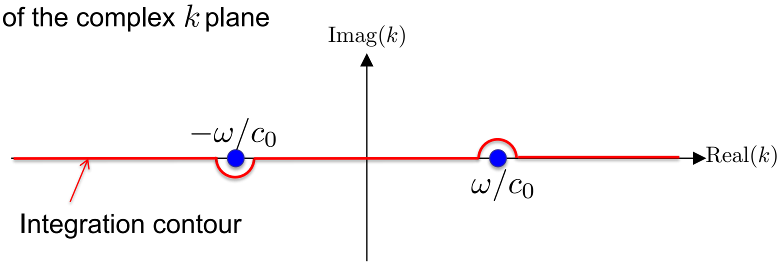
Fourier transforms

Doing the inverse Fourier Transform in space leads to:

$$\tilde{p}(x,\omega) = \frac{1}{4\pi^2 c_0^2} \int_{-\infty}^{\infty} \frac{e^{-jkx}}{k^2 - \omega^2/c_0^2}\, dk$$

Use of the complex k plane



The result is $\tilde{p}(x,\omega) = \dfrac{e^{-j\omega|x|/c_0}}{4\pi j c_0 \omega}$

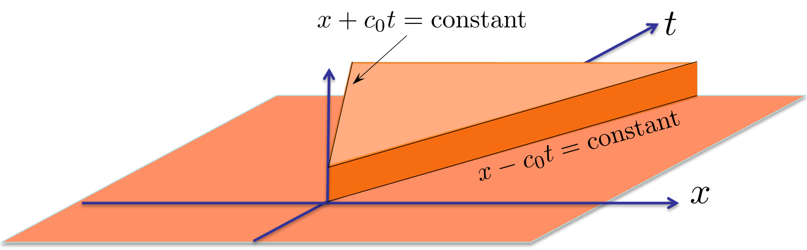
Fourier transforms

Doing the inverse Fourier Transform in time leads to:

$$p(x,t) = \frac{1}{4\pi j c_0} \int_{-\infty}^{\infty} \frac{e^{j\omega(t-|x|/c_0)}}{\omega}\, d\omega$$

$$= \frac{1}{2c_0} H(t - |x|/c_0)$$

H is the Heaviside step function



Sources of sound and energy

p 41

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p 42

Source of sound

$$\left\{\begin{array}{l}\frac{1}{c_0^2}\frac{\partial p'}{\partial t}+\rho_0\vec{\nabla}.\vec{u'}=0\\ \rho_0\frac{\partial \vec{u'}}{\partial t}+\vec{\nabla}p'=0\end{array}\right. \Rightarrow \left\{\begin{array}{l}\frac{1}{c_0^2}\frac{\partial p'}{\partial t}+\rho_0\vec{\nabla}.\vec{u'}=Q_m\\ \rho_0\frac{\partial \vec{u'}}{\partial t}+\vec{\nabla}p'=\vec{f}\end{array}\right.$$

Wave equation with sources

$$\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}p'-\Delta p'=\frac{\mathrm{d}Q_m}{\mathrm{d}t}-\vec{\nabla}\vec{f}$$

Monopole =
Non-stationary mass injection

Dipole =
Non-uniform force field on fluid

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Source of sound

Monopole = Non-stationary mass injection

Example: The pulsating sphere

For symmetry reasons, the pulsating sphere will produce a spherical field

$$p = \frac{A_0}{r}\cos(\omega t - k_0 r)$$

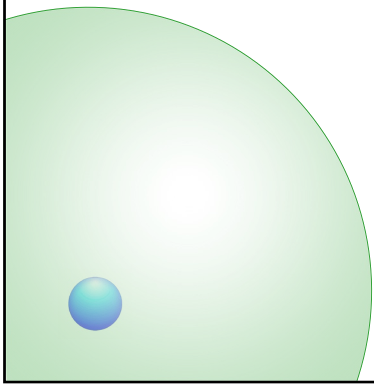
The radius is $a = a_0 + \hat{a}e^{\mathrm{j}\omega t}$

The mass flow rate is

$$Q_m = \rho_0(\pi a_0^2)\mathrm{j}\omega\hat{a}$$

$\hat{u}_{r0}$

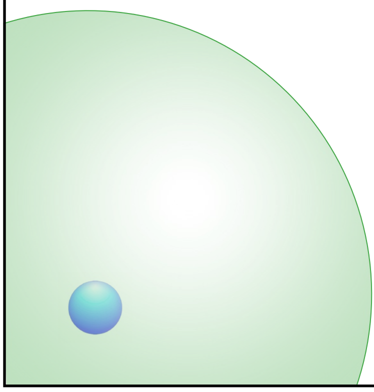
$$\hat{u}_r = \frac{A_0}{\rho_0 c_0 r}\left(1 + \frac{1}{\mathrm{j}k_0 r}\right)$$



Source of sound

$$k_0 a_0 \ll 1 \quad \rightarrow \quad \hat{u}_{r0} = \frac{Q_m}{\rho_0 \pi a_0^2} = \frac{A_0}{\rho_0 a_0^2 \mathrm{j}\omega} \qquad A_0 = \frac{\mathrm{j}\omega Q_m}{\pi}$$

$$p = \frac{\mathrm{j}\omega Q_m}{\pi r} e^{\mathrm{j}(\omega t - k_0 r)}$$



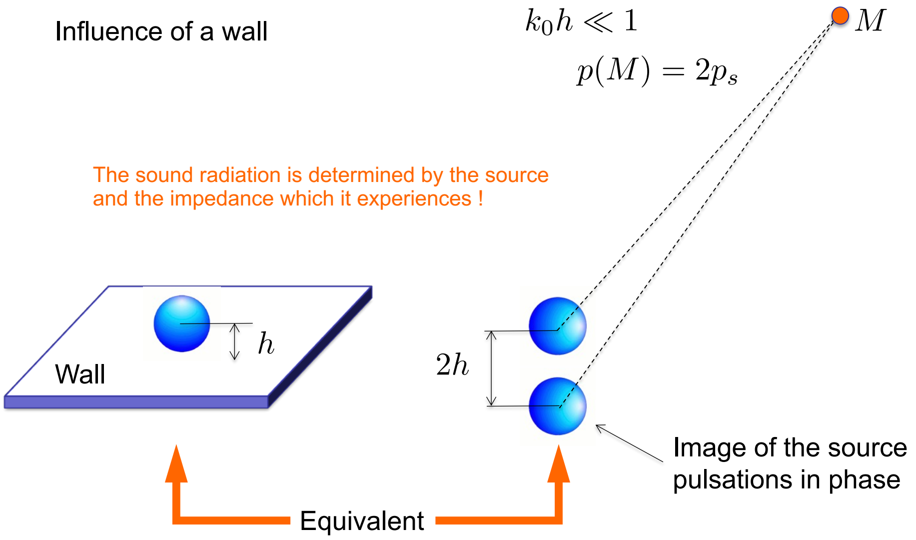
Source of sound

Influence of a wall

$k_0 h \ll 1$

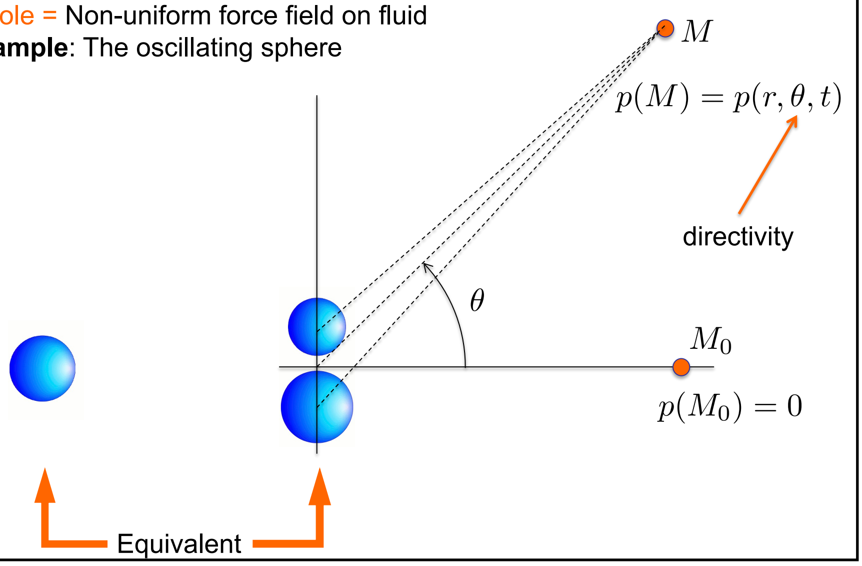
$p(M) = 2p_s$

The sound radiation is determined by the source and the impedance which it experiences !

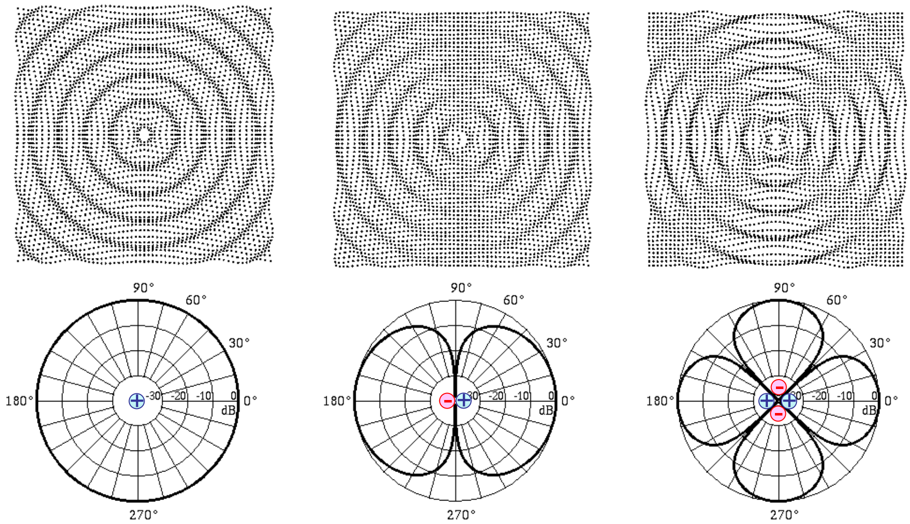


Source of sound

Dipole = Non-uniform force field on fluid
Example: The oscillating sphere



Source of sound



Acoustical energy

$$\left[\frac{1}{c_0^2} \frac{\partial p'}{\partial t} + \rho_0 \vec{\nabla} . \vec{u'} = Q_m \right] \frac{p'}{\rho_0}$$

$$\left[\rho_0 \frac{\partial \vec{u'}}{\partial t} + \vec{\nabla} p' = \vec{f} \right] . \vec{u'}$$

$$\frac{\partial E}{\partial t} + \vec{\nabla} \vec{I} = \vec{u'} . \vec{f} + p' \frac{Q_m}{\rho_0}$$

Energy $E = \frac{1}{2} \rho_0 (\vec{u'})^2 + \frac{1}{2} \frac{(p')^2}{\rho_0 c_0^2}$

Power injected by volume sources (per unit volume)

Intensity $\vec{I} = p' \vec{u'}$

Power injected by force sources (per unit volume)

Acoustical energy

We deduced an equation for quadratic functions of perturbations from linear approximation

Acoustical energy

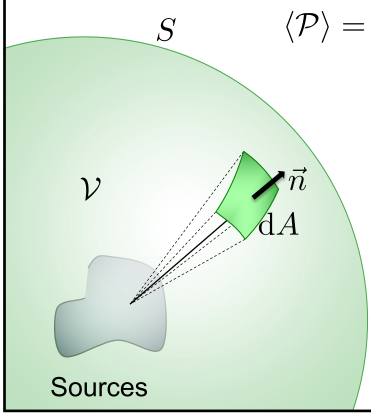
$$\frac{\partial E}{\partial t} + \vec{\nabla}.\vec{I} = \mathcal{P}sources$$

$$\langle \mathcal{P} \rangle = \iint_S \mathrm{Real}\langle \vec{n}.\vec{u}p \rangle \mathrm{d}A$$

$$\iint_S \langle \vec{n}.\vec{I} \rangle \mathrm{d}A = \iiint_{\mathcal{V}} \langle \mathcal{P}sources \rangle \, \mathrm{dV}$$

Intensity flux

$$\langle g \rangle = \frac{1}{T}\int_0^T g(t)\mathrm{d}t$$

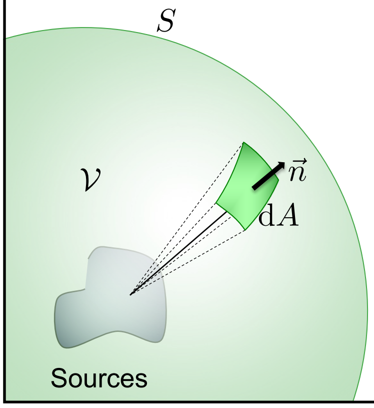


Source of sound

Acoustical energy
for spherical waves in time harmonic motion

$$\vec{n}.\vec{u} = u_r = \frac{p}{\rho_0 c_0}\left(1 + \frac{1}{\mathrm{j}k_0 r}\right)$$

$$\vec{n}.\vec{I} = \frac{p^2}{\rho_0 c_0} \qquad \text{for } r \gg \lambda$$



$$\langle \mathcal{P} \rangle = \frac{\pi r^2}{2}\frac{|p|^2}{\rho_0 c_0}$$

Acoustical power

Source of sound

Acoustical energy
in time harmonic motion

More generally $\vec{n}.\vec{u} = u_n = Z(\omega)p$

$$\langle \mathcal{P} \rangle = \iint_S \frac{\omega}{2\pi}\int_0^{2\pi/\omega} \mathrm{Real}\left(\hat{u}_n e^{\mathrm{j}\omega t}\right) \mathrm{Real}\left(\hat{p} e^{\mathrm{j}\omega t}\right) \mathrm{d}t \;\mathrm{d}A$$

$$= \iint_S \frac{1}{4}\left(\hat{p}\hat{u}_n^* + \hat{p}^*\hat{u}_n\right)\,\mathrm{d}A$$

$$= \iint_S \frac{1}{2}\mathrm{Real}\left(\hat{p}^*\hat{u}_n\right)\,\mathrm{d}A$$

$$\langle \mathcal{P} \rangle = \iint_S \frac{1}{2}\mathrm{Real}(Z)|\hat{u}_n|^2\,\mathrm{d}A$$

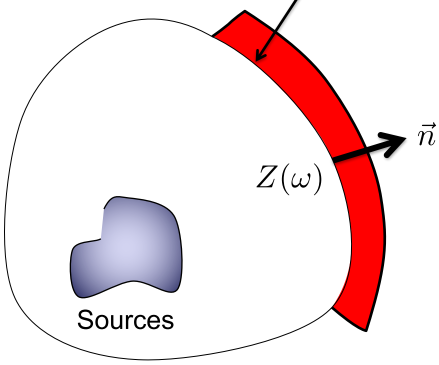
Acoustical power

Source of sound

Acoustical energy

$$\langle \mathcal{P} \rangle = \iint_S \frac{1}{2} \mathrm{Real}(Z) |\hat{u}_n|^2 \, \mathrm{d}A$$

$$\vec{n}.\vec{u} = u_n = Z(\omega) p$$



For a wall to be passive (absorbing energy), the real part of the impedance had to be positive (normal $\vec{n}$ in the wall)

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Green's function

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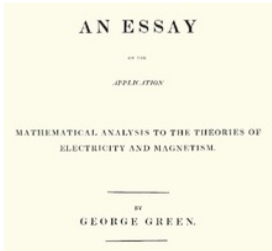
Green's functions

The Green's function $G(\vec{x},t|\vec{y},\tau)$ is the pulse response of wave equation

$$\frac{\partial^2 G}{\partial t^2} - c_0^2 \Delta G = \delta(\vec{x}-\vec{y})\delta(t-\tau)$$

$G(\vec{x},t|\vec{y},\tau)$ is the response at the observation point $\vec{x}$ at time t of a source at point $\vec{y}$ at time τ

George Green
(1793 –1841)


AN ESSAY

APPLICATION

MATHEMATICAL ANALYSIS TO THE THEORIES OF ELECTRICITY AND MAGNETISM.

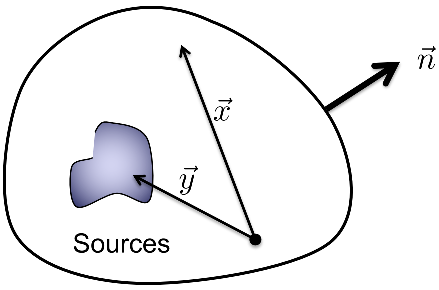
GEORGE GREEN.

Green's functions

To find the Green's function it is necessary to add to the equation:

$$\frac{\partial^2 G}{\partial t^2} - c_0^2 \Delta G = \delta(\vec{x}-\vec{y})\delta(t-\tau)$$

the boundary conditions on a surface S enclosing the volume V in which $\vec{x}$ and $\vec{y}$ are localized



$\vec{n}.\vec{\nabla}G + b\,G = c$ Robin condition

$\vec{n}.\vec{\nabla}G = c$ Neumann condition

$G = a$ Dirichlet condition

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Green's functions

Solution of the wave equation with sources: $$\frac{\partial^2 p}{\partial t^2} - c_0^2 \Delta p = q(\vec{y},\tau)$$

$$p(\vec{x},t) = \int_0^t \iiint_V q(\vec{y},\tau) G(\vec{x},t|\vec{y},\tau) \mathrm{d}^3\vec{y}\ \mathrm{d}\tau$$

Effect of sources

$$+ c_0^2 \int_0^t \iint_S (G\vec{\nabla}p - p\vec{\nabla}G).\vec{n}_y\ \mathrm{d}^2\vec{y}\ \mathrm{d}\tau$$

Effect of boundary condition

$$- \left[\iiint_V (p(\vec{y},\tau)\frac{\partial G}{\partial \tau} - G\frac{\partial p(\vec{y},\tau)}{\partial \tau})\ \mathrm{d}^3\vec{y} \right]_{\tau=0}$$

Effect of initial condition

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Green's functions

In the frequency domain:

$$\hat{p}(\vec{x}) = \iiint_V \hat{q}(\vec{y})\ \hat{G}(\vec{x}|\vec{y})\ \mathrm{d}^3\vec{y}$$

Effect of sources

$$+ \iint_S (\hat{G}\vec{\nabla}\hat{p} - \hat{p}\vec{\nabla}\hat{G}).\vec{n}_y\ \mathrm{d}^2\vec{y}$$

Effect of boundary condition

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Green's functions

Free field Green's functions:

	1D	3D
$\Delta G + k^2 G = \delta(\vec{x} - \vec{y})$	$\frac{\mathrm{j}}{2k} e^{\mathrm{j}k\|X\|}$	$\frac{e^{-\mathrm{j}kR}}{4\pi R}$
$\frac{\partial^2 G}{\partial t^2} - c_0^2 \Delta G = \delta(\vec{x} - \vec{y})\delta(t - \tau)$	$\frac{1}{2c_0} H(T - \|X\|/c_0)$	$\frac{\delta(T - R/c_0)}{4\pi c_0 R}$

$T = t - \tau \qquad X = x - y \qquad R = \|\vec{x} - \vec{y}\|$

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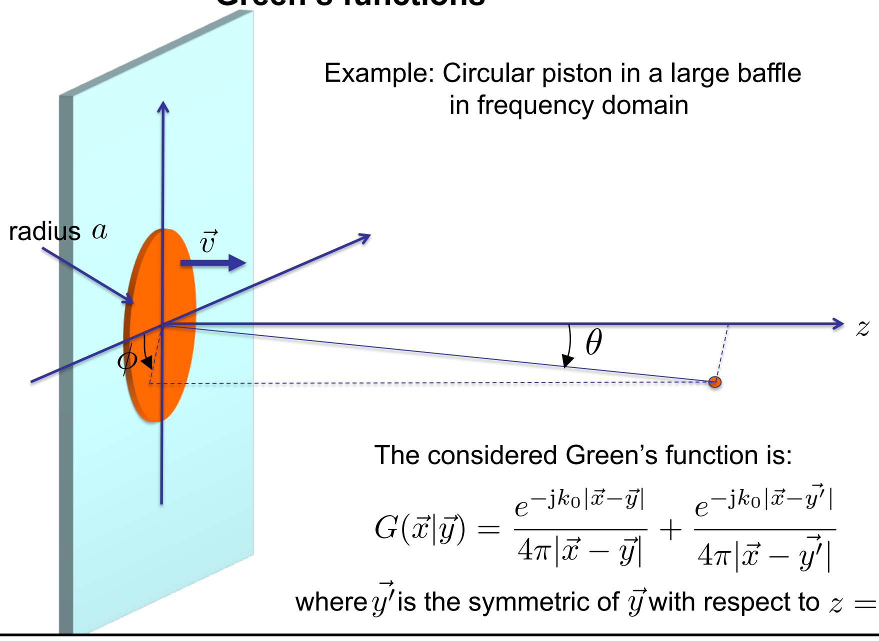
Green's functions

Example: Circular piston in a large baffle
in frequency domain



The considered Green's function is:

$$G(\vec{x}|\vec{y}) = \frac{e^{-\mathrm{j}k_0|\vec{x}-\vec{y}|}}{4\pi|\vec{x}-\vec{y}|} + \frac{e^{-\mathrm{j}k_0|\vec{x}-\vec{y}'|}}{4\pi|\vec{x}-\vec{y}'|}$$

where $\vec{y}'$ is the symmetric of $\vec{y}$ with respect to $z = 0$

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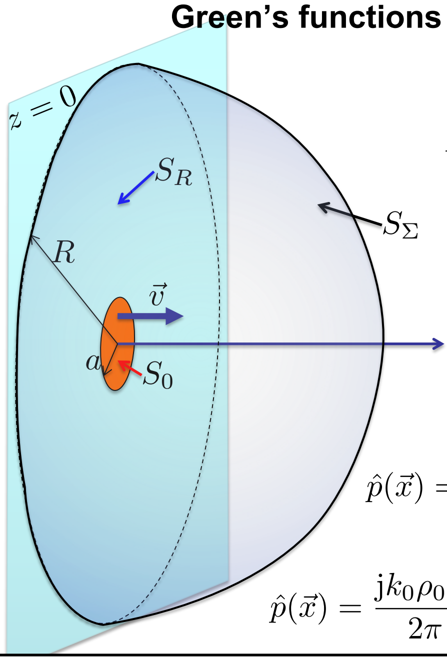
Green's functions



no sources in the domain

$$\hat{p}(\vec{x}) = \iiint_V \hat{q}(\vec{y})\,\hat{G}(\vec{x}|\vec{y})\,\mathrm{d}^3\vec{y}$$

$$+ \iint_S (\hat{G}\vec{\nabla}\hat{p} - \hat{p}\vec{\nabla}\hat{G}).\vec{n}_y\,\mathrm{d}^2\vec{y}$$

$\hat{G}$ symmetric / $z=0$
+
Sommerfeld

$$\hat{p}(\vec{x}) = -\iint_{S_0} \hat{G}\frac{\partial p}{\partial z}\,r\mathrm{d}r\,\mathrm{d}\phi$$

$$\hat{p}(\vec{x}) = \frac{\mathrm{j}k_0\rho_0 c_0}{2\pi}\iint_{S_0}\frac{e^{-\mathrm{j}k_0|\vec{x}-\vec{y_0}|}}{|\vec{x}-\vec{y_0}|}v(\vec{y_0})\,r\mathrm{d}r\,\mathrm{d}\phi$$

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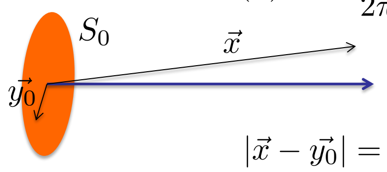
Green's functions

$$\hat{p}(\vec{x}) = \frac{\mathrm{j}k_0\rho_0 c_0}{2\pi}\iint_{S_0}\frac{e^{-\mathrm{j}k_0|\vec{x}-\vec{y_0}|}}{|\vec{x}-\vec{y_0}|}v(\vec{y_0})\,r\mathrm{d}r\,\mathrm{d}\phi$$



In the far field $x \gg y_0$

$$|\vec{x}-\vec{y_0}| = \sqrt{x^2+y_0^2-2\vec{x}.\vec{y_0}} \simeq r - \frac{\vec{x}.\vec{y_0}}{r}$$

$$\hat{p}(\vec{x}) = \mathrm{j}k_0\rho_0 c_0 v\frac{e^{-\mathrm{j}k_0 r}}{2\pi r}\int_0^{2\pi}\int_0^a e^{\mathrm{j}k_0 r_0\sin\theta\cos(\phi_0-\phi)}\,r_0\mathrm{d}r_0\,\mathrm{d}\phi_0$$

$$\hat{p}(\vec{x}) = \underbrace{\mathrm{j}2\pi a^2 k_0\rho_0 c_0 v\frac{e^{-\mathrm{j}k_0 r}}{4\pi r}}_{\text{like monopole}}\;\underbrace{\frac{2J_1(k_0 a\sin\theta)}{k_0 a\sin\theta}}_{\text{directivity}}$$

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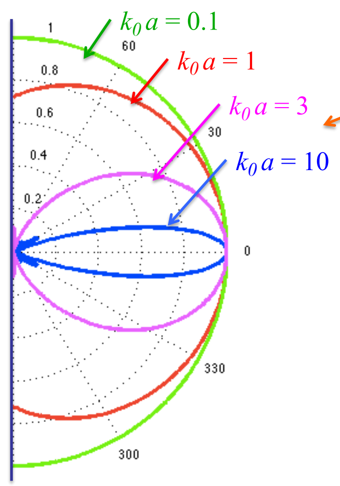
p 63

Green's functions

$$\hat{p}(\vec{x}) = \mathrm{j}2\pi a^2 k_0 \rho_0 c_0 v \frac{e^{-\mathrm{j}k_0 r}}{4\pi r} \underbrace{\frac{2J_1(k_0 a \sin\theta)}{k_0 a \sin\theta}}_{\text{directivity}}$$



John William Strutt
Lord Rayleigh
(1842 - 1919)

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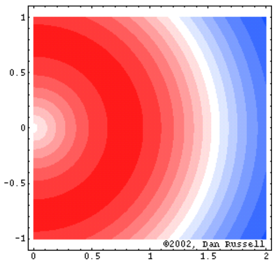
p 64

Green's functions

$$\hat{p}(\vec{x}) = \mathrm{j}2\pi a^2 k_0 \rho_0 c_0 v \frac{e^{-\mathrm{j}k_0 r}}{4\pi r} \frac{2J_1(k_0 a \sin\theta)}{k_0 a \sin\theta}$$

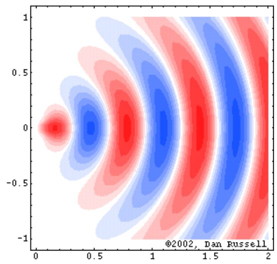
Low frequency

$k_0 a < 1$



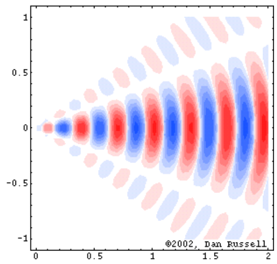
Medium frequency

$k_0 a > 1$



High frequency

$k_0 a \gg 1$



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Green's functions

$$\hat{p}(\vec{x}) = \frac{\mathrm{j}k_0\rho_0 c_0}{2\pi}\iint_{S_0}\frac{e^{-\mathrm{j}k_0|\vec{x}-\vec{y_0}|}}{|\vec{x}-\vec{y_0}|}v(\vec{y_0})\, r\mathrm{d}r\,\mathrm{d}\phi$$

By integration of $\hat{p}(\vec{x})$ on the surface S_0, the mean pressure on the surface S_0 divided by the piston velocity can be written

Struve function

Bessel function

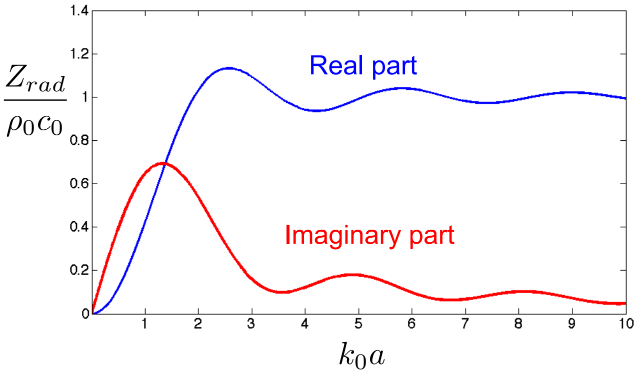
$$Z_{rad} = \rho_0 c_0\left(1 - \frac{J_1(2k_0a)}{k_0a} + \mathrm{j}\frac{S_1(2k_0a)}{k_0a}\right)$$

For $k_0a \ll 1$ $\quad Z_{rad} = \rho_0 c_0\left(\frac{1}{2}(k_0a)^2 + \mathrm{j}\frac{8k_0a}{3\pi}\right)$

Rayleigh radiation impedance

Green's functions

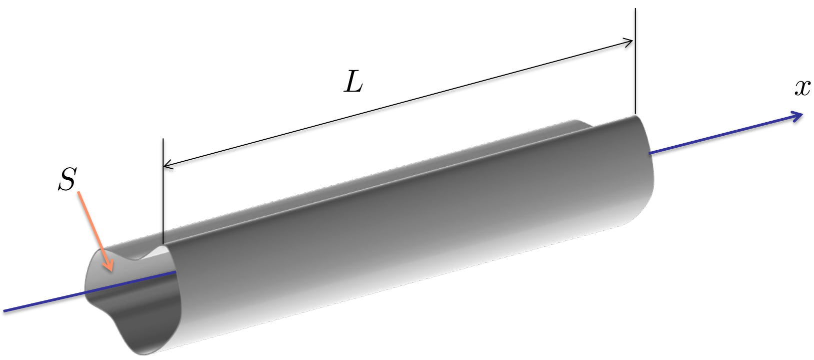
$$Z_{rad} = \rho_0 c_0\left(1 - \frac{J_1(2k_0a)}{k_0a} + \mathrm{j}\frac{S_1(2k_0a)}{k_0a}\right)$$



For $k_0a \ll 1$ $\quad Z_{rad} = \rho_0 c_0\left(\frac{1}{2}(k_0a)^2 + \mathrm{j}\frac{8k_0a}{3\pi}\right)$

Low frequencies propagation in duct (1D)

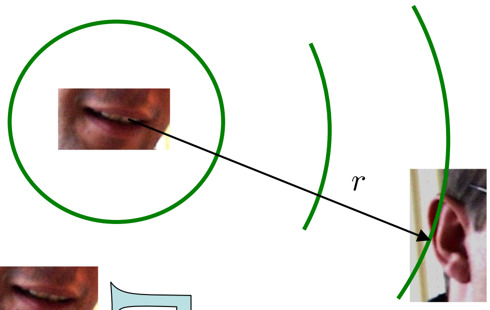
Propagation in ducts (1D)



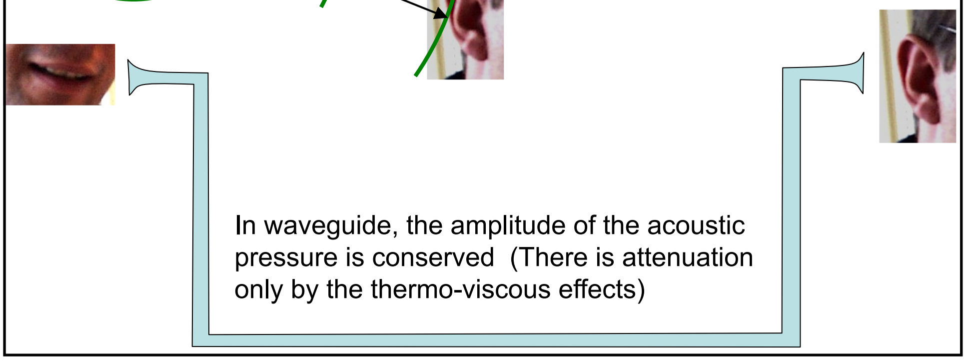
$$p(x,\omega) = \left(p^+ \mathrm{e}^{-\mathrm{j}k_0 x} + p^- \mathrm{e}^{\mathrm{j}k_0 x}\right) \mathrm{e}^{\mathrm{j}\omega t}$$

$$u(x,\omega) = \frac{1}{\rho_0 c_0} \left(p^+ \mathrm{e}^{-\mathrm{j}k_0 x} - p^- \mathrm{e}^{\mathrm{j}k_0 x}\right) \mathrm{e}^{\mathrm{j}\omega t}$$

Propagation in ducts (1D)



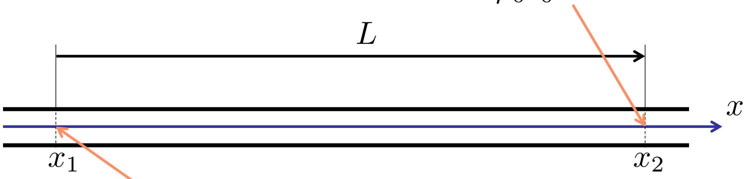
In free field, the acoustic pressure in proportional to $1/r$ => There is a geometric attenuation of - 6 dB when the distance to the source is doubled



In waveguide, the amplitude of the acoustic pressure is conserved (There is attenuation only by the thermo-viscous effects)

Propagation in ducts (1D)

$$p_2 = p_0^+ \mathrm{e}^{-\mathrm{j}k_0x_2} + p_0^- \mathrm{e}^{\mathrm{j}k_0x_2}$$
$$u2 = \frac{1}{\rho_0 c_0}\left(p_0^+ \mathrm{e}^{-\mathrm{j}k_0x_2} - p_0^- \mathrm{e}^{\mathrm{j}k_0x_2}\right)$$

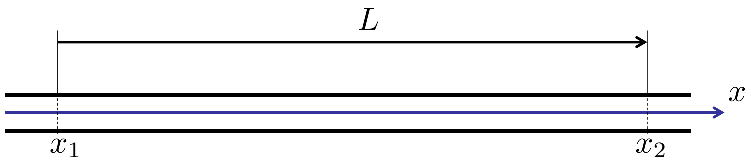


$$p_1 = p_0^+ \mathrm{e}^{-\mathrm{j}k_0x_1} + p_0^- \mathrm{e}^{\mathrm{j}k_0x_1}$$
$$u1 = \frac{1}{\rho_0 c_0}\left(p_0^+ \mathrm{e}^{-\mathrm{j}k_0x_1} - p_0^- \mathrm{e}^{\mathrm{j}k_0x_1}\right)$$

Propagation in ducts (1D)

$$p_2 = p_2^+ + p_2^-$$

$$u_2 = \frac{1}{\rho_0 c_0}\left(p_2^+ - p_2^-\right)$$



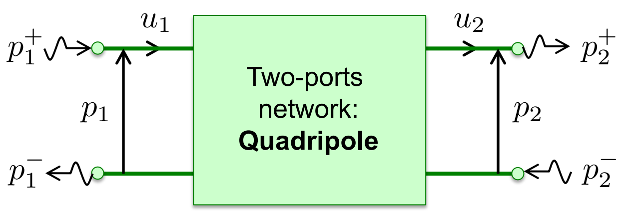
$$p_1 = p_1^+ + p_1^-$$

$$u_1 = \frac{1}{\rho_0 c_0}\left(p_1^+ - p_1^-\right)$$

$$p_2^+ = p_1^+ \mathrm{e}^{-\mathrm{j}k_0(x_2 - x1)} = p_1^+ \mathrm{e}^{-\mathrm{j}k_0 L}$$

$$p_1^- = p_2^- \mathrm{e}^{-\mathrm{j}k_0(x_2 - x1)} = p_2^- \mathrm{e}^{-\mathrm{j}k_0 L}$$

Propagation in ducts (1D)



Equivalence with electrical circuits

p 73

Propagation in ducts (1D)

The quadripole can be represented by a 2 × 2 matrix.

There are several ways to do:

Transmission Matrix

$$\begin{pmatrix} p_2 \\ u_2 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} p_1 \\ u_1 \end{pmatrix}$$

Impedance Matrix

$$\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

Scattering Matrix

$$\begin{pmatrix} p_1^- \\ p_2^+ \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} p_1^+ \\ p_2^- \end{pmatrix}$$

Transfer Matrix + …

$$\begin{pmatrix} p_2^+ \\ p_2^- \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \begin{pmatrix} p_1^+ \\ p_1^- \end{pmatrix}$$

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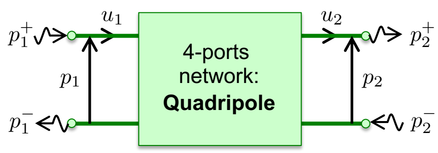
Propagation in ducts (1D)



In general those matrices can have proprieties associated with:

Reciprocity: A network is said to be reciprocal if the pressure appearing at port 2 due to a velocity applied at port 1 is the same as the pressure appearing at port 1 when the same velocity is applied to port 2.

$AD - BC = 1$ $\quad Z_{12} = -Z_{21}$ $\quad S_{21} = S_{12}$

Symmetry: A network is said to be symmetric if the port 1 and the port 2 can reversed.

$A = D$ $\quad Z_{11} = -Z_{22}$ $\quad S_{11} = S_{22}$

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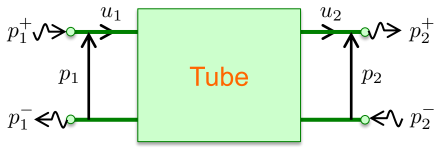
Propagation in ducts (1D)



$$p_1^- = p_2^- \mathrm{e}^{-\mathrm{j}k_0 L}$$
$$p_2^+ = p_1^+ \mathrm{e}^{-\mathrm{j}k_0 L}$$

Scattering Matrix

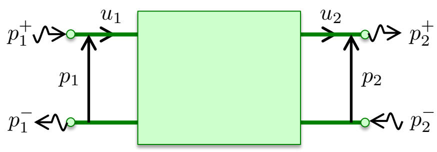
$$\begin{pmatrix} p_1^- \\ p_2^+ \end{pmatrix} = \begin{pmatrix} R & t \\ T & r \end{pmatrix} \begin{pmatrix} p_1^+ \\ p_2^- \end{pmatrix}$$

$$T = t = \mathrm{e}^{-\mathrm{j}k_0 L}$$
$$R = r = 0$$

$$\begin{pmatrix} p_1^- \\ p_2^+ \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{e}^{-\mathrm{j}k_0 L} \\ \mathrm{e}^{-\mathrm{j}k_0 L} & 0 \end{pmatrix} \begin{pmatrix} p_1^+ \\ p_2^- \end{pmatrix}$$

Scattering Matrix

Propagation in ducts (1D)



Scattering Matrix

$$\begin{pmatrix} p_1^- \\ p_2^+ \end{pmatrix} = \begin{pmatrix} R & t \\ T & r \end{pmatrix} \begin{pmatrix} p_1^+ \\ p_2^- \end{pmatrix}$$

Link to the
Transfer Matrix

$$\begin{aligned} p_1^- &= Rp_1^+ + tp_2^- \\ p_2^+ &= Tp_1^+ + rp_2^- \end{aligned}$$

$$\begin{aligned} tp_2^- &= -Rp_1^+ + p_1^- \\ p_2^+ - rp_2^- &= Tp_1^+ \end{aligned}$$

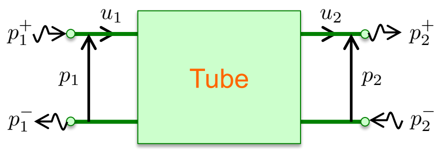
$$\begin{pmatrix} 0 & t \\ 1 & -r \end{pmatrix} \begin{pmatrix} p_2^+ \\ p_2^- \end{pmatrix} = \begin{pmatrix} -R & 1 \\ T & 0 \end{pmatrix} \begin{pmatrix} p_1^+ \\ p_1^- \end{pmatrix}$$

Transfer Matrix

$$\begin{pmatrix} p_2^+ \\ p_2^- \end{pmatrix} = \begin{pmatrix} T - R\,t^{-1}\,r & t^{-1}\,r \\ -Rt^{-1} & t^{-1} \end{pmatrix} \begin{pmatrix} p_1^+ \\ p_1^- \end{pmatrix}$$

p 77

Propagation in ducts (1D)



Transfer Matrix

$$\begin{pmatrix} p_2^+ \\ p_2^- \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \begin{pmatrix} p_1^+ \\ p_1^- \end{pmatrix}$$

$$\begin{pmatrix} p_2^+ \\ p_2^- \end{pmatrix} = \begin{pmatrix} T - R\,t^{-1}\,r & t^{-1}\,r \\ -Rt^{-1} & t^{-1} \end{pmatrix} \begin{pmatrix} p_1^+ \\ p_1^- \end{pmatrix}$$

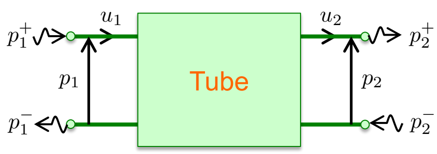
$$\begin{pmatrix} p_2^+ \\ p_2^- \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{-\mathrm{j}k_0 L} & 0 \\ 0 & \mathrm{e}^{\mathrm{j}k_0 L} \end{pmatrix} \begin{pmatrix} p_1^+ \\ p_1^- \end{pmatrix}$$

Transfer Matrix

77

p 78

Propagation in ducts (1D)



Transfer Matrix

$$\begin{pmatrix} p_2^+ \\ p_2^- \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{-\mathrm{j}k_0 L} & 0 \\ 0 & \mathrm{e}^{\mathrm{j}k_0 L} \end{pmatrix} \begin{pmatrix} p_1^+ \\ p_1^- \end{pmatrix}$$

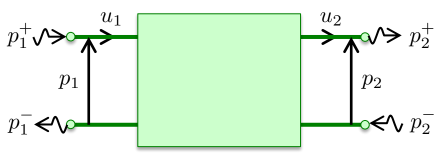
$$p_2 = p_2^+ + p_2^- = \mathrm{e}^{-\mathrm{j}k_0 L} p_1^+ + \mathrm{e}^{\mathrm{j}k_0 L} p_1^-$$
$$u_2 = Z_c^{-1}(p_2^+ - p_2^-) = Z_c^{-1}\left(\mathrm{e}^{-\mathrm{j}k_0 L} p_1^+ - \mathrm{e}^{\mathrm{j}k_0 L} p_1^-\right)$$

$$p_1 = p_1^+ + p_1^-$$
$$u_1 = Z_c^{-1}(p_1^+ - p_1^-)$$

$$p_1^+ = \frac{1}{2}(p_1 + Z_c u_1)$$
$$p_1^- = \frac{1}{2}(p_1 - Z_c u_1)$$

78

Propagation in ducts (1D)



Transfer Matrix

$$\begin{pmatrix} p_2^+ \\ p_2^- \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \begin{pmatrix} p_1^+ \\ p_1^- \end{pmatrix}$$

Link to the Transmission Matrix

$Z_c = \rho_0 c_0$

$$\begin{pmatrix} p_1 \\ u_1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ Z_c^{-1} & -Z_c^{-1} \end{pmatrix} \begin{pmatrix} p_1^+ \\ p_1^- \end{pmatrix}$$

$$\begin{pmatrix} p_2 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ Z_c^{-1} & -Z_c^{-1} \end{pmatrix} \begin{pmatrix} p_2^+ \\ p_2^- \end{pmatrix}$$

$$\begin{pmatrix} p_2 \\ u_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 \\ Z_c^{-1} & -Z_c^{-1} \end{pmatrix} \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ Z_c^{-1} & -Z_c^{-1} \end{pmatrix}^{-1}}_{\text{Transmission Matrix}} \begin{pmatrix} p_1 \\ u_1 \end{pmatrix}$$

Propagation in ducts (1D)

$$p_2 = \frac{1}{2}\left(\mathrm{e}^{-\mathrm{j}k_0 L} + \mathrm{e}^{\mathrm{j}k_0 L}\right) p_1 + \frac{1}{2}\left(\mathrm{e}^{-\mathrm{j}k_0 L} - \mathrm{e}^{\mathrm{j}k_0 L}\right) Z_c u_1$$

$$Z_c u_2 = \frac{1}{2}\left(\mathrm{e}^{-\mathrm{j}k_0 L} - \mathrm{e}^{\mathrm{j}k_0 L}\right) p_1 + \frac{1}{2}\left(\mathrm{e}^{-\mathrm{j}k_0 L} + \mathrm{e}^{\mathrm{j}k_0 L}\right) Z_c u_1$$

$$p_2 = \cos(k_0 L) p_1 - \mathrm{j}\sin(k_0 L) Z_c u_1$$

$$Z_c u_2 = -\mathrm{j}\sin(k_0 L) p_1 + \cos(k_0 L) Z_c u_1$$

$$\begin{pmatrix} p_2 \\ u_2 \end{pmatrix} = \begin{pmatrix} \cos(k_0 L) & -\mathrm{j}Z_c \sin(k_0 L) \\ -\mathrm{j}Z_c^{-1}\sin(k_0 L) & \cos(k_0 L) \end{pmatrix} \begin{pmatrix} p_1 \\ u_1 \end{pmatrix}$$

Transmission Matrix

p 81

Propagation in ducts (1D)

$$Z_1 = \frac{p_1}{u_1} = \frac{\cos(k_0 L) p_2 + \mathrm{j} Z_c \sin(k_0 L) u_2}{\mathrm{j} Z_c^{-1} \sin(k_0 L) p_2 + \cos(k_0 L) u_2}$$

$$Z_1/Z_c = \frac{Z_2/Z_c + \mathrm{j}\tan(k_0 L)}{1 + \mathrm{j} Z_2/Z_c \tan(k_0 L)}$$

Transport of impedance

$$R_1 = \frac{p_1^-}{p_1^+} = R_2\, \mathrm{e}^{-2\mathrm{j} k_0 L}$$

Transport of reflection coefficient

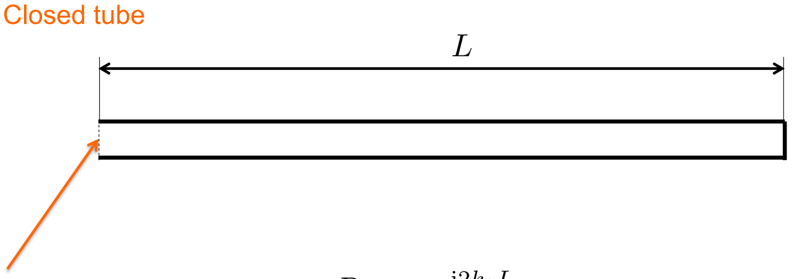
81

p 82

Propagation in ducts (1D)

$$Z_1/Z_c = \frac{Z_2/Z_c + \mathrm{j}\tan(k_0 L)}{1 + \mathrm{j} Z_2/Z_c \tan(k_0 L)} \qquad L = x_2 - x_1$$

Closed tube



$u_2 = 0$

$Z_2 = \infty$

$R_2 = 1$

$$R_1 = \mathrm{e}^{-\mathrm{j}2k_0 L}$$

$$Z_1/Z_c = \frac{1}{\mathrm{j}\tan(k_0 L)}$$

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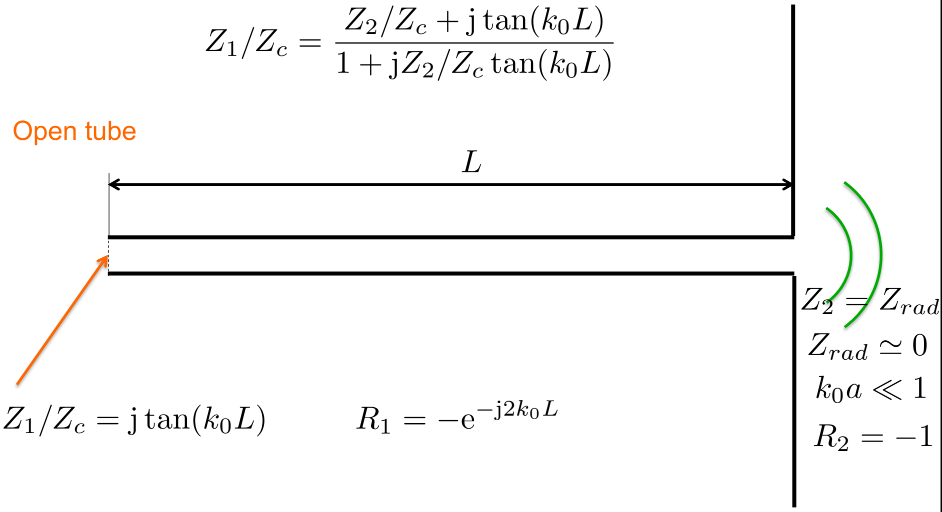
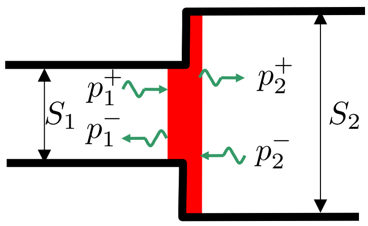
Propagation in ducts (1D)

$$Z_1/Z_c = \frac{Z_2/Z_c + \mathrm{j}\tan(k_0 L)}{1 + \mathrm{j}Z_2/Z_c \tan(k_0 L)}$$



Propagation in ducts (1D)



Area expansion $k_0 a \ll 1$

Continuity of pressure $p_1 = p_2$

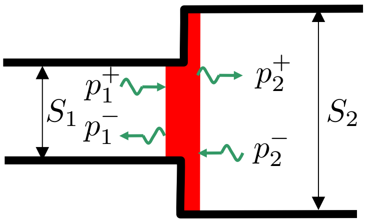
$$p_1^+ + p_1^- = p_2^+ + p_2^-$$

Continuity of acoustic flow rate $S_1 u_1 = S_2 u_2$

$$S_1(p_1^+ - p_1^-) = S_2(p_2^+ - p_2^-)$$

Propagation in ducts (1D)

Area expansion



Scattering Matrix

$$\begin{pmatrix} p_1^- \\ p_2^+ \end{pmatrix} = \begin{pmatrix} R & t \\ T & r \end{pmatrix} \begin{pmatrix} p_1^+ \\ p_2^- \end{pmatrix}$$

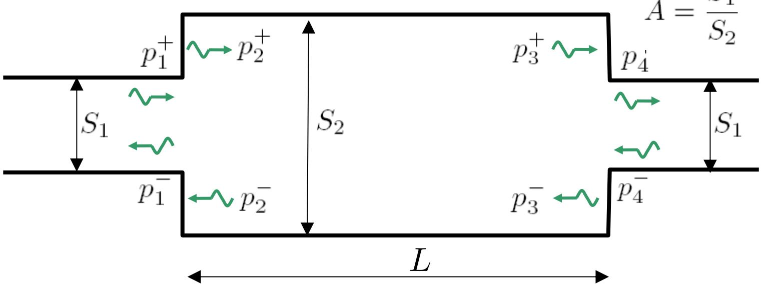
$$R = -\frac{S_2/S_1 - 1}{S_2/S_1 + 1} \qquad t = \frac{2S_2/S_1}{S_2/S_1 + 1}$$

$$T = \frac{2}{S_2/S_1 + 1} \qquad r = \frac{S_2/S_1 - 1}{S_2/S_1 + 1}$$

Propagation in ducts (1D)

Expansion chamber

$$A = \frac{S_1}{S_2}$$



$$p_1^+ + p_1^- = p_2^+ + p_2^- \qquad p_4^+ + p_4^- = p_3^+ + p_3^-$$

$$A\left(p_1^+ - p_1^-\right) = p_2^+ - p_2^- \qquad A\left(p_4^+ - p_4^-\right) = p_3^+ - p_3^-$$

$$p_3^+ = p_2^+ \exp(-\mathrm{j}k_0 L)$$

$$p_3^- = p_2^- \exp(\mathrm{j}k_0 L)$$

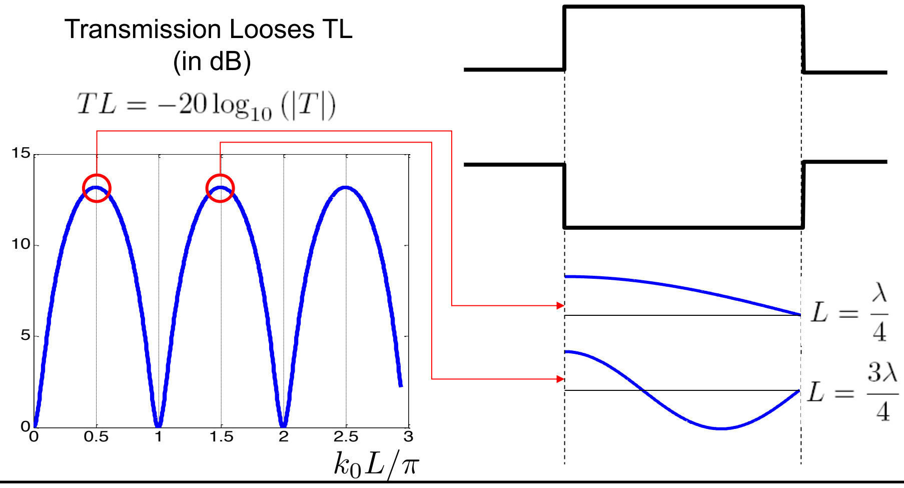
Propagation in ducts (1D)

Expansion chamber

$$T = \frac{2A\exp(\mathrm{j}k_0L)}{2A\cos(k_0L) + \mathrm{j}(1+A^2)\sin(k_0L)}$$

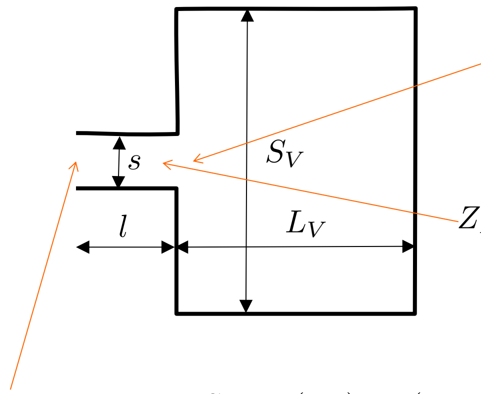
Transmission Looses TL
(in dB)

$$TL = -20\log_{10}(|T|)$$



Propagation in ducts (1D)

Resonator



$$Z_V/(\rho_0c_0) = \frac{-\mathrm{j}}{\tan(k_0L_V)}$$

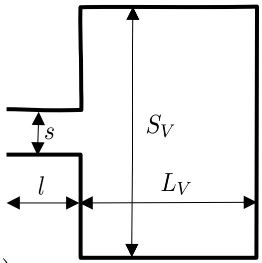
$$Z_N/(\rho_0c_0) = \frac{-\mathrm{j}s}{S_V\tan(k_0L_V)}$$

$$Z_R/(\rho_0c_0) = \mathrm{j}\frac{S_V\tan(k_0l)\tan(k_0L_V) - s}{S_V\tan(k_0L_V) + s\tan(k_0l)}$$

Propagation in ducts (1D)

Resonator

$$k_0 l \ll 1$$
$$k_0 L_V \ll 1$$



$$Z_R/(\rho_0 c_0) = \mathrm{j}\frac{S_V \tan(k_0 l)\tan(k_0 L_V) - s}{S_V \tan(k_0 L_V) + s\tan(k_0 l)}$$

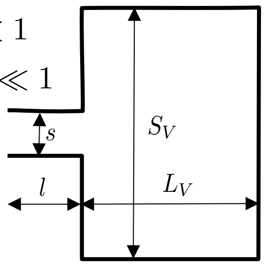
$$Z_R/(\rho_0 c_0) = \mathrm{j}\frac{k_0^2 l S_V L_V - s}{k_0(S_V L_V + sl)} \qquad V = S_V L_V \gg sl$$

$$Z_R/(\rho_0 c_0) = \mathrm{j}(k_0 l - \frac{s}{k_0 V})$$

Propagation in ducts (1D)

Resonator

$$k_0 l \ll 1$$
$$k_0 L_V \ll 1$$



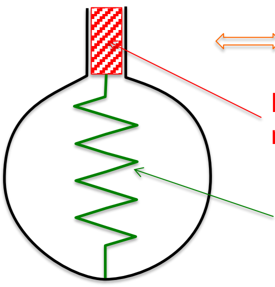
$$Z_R/(\rho_0 c_0) = \mathrm{j}(k_0 l - \frac{s}{k_0 V})$$

$$Z_R = 0 \Rightarrow k_0^2 = \frac{s}{lV}$$

$$\omega_R = c_0\sqrt{\frac{s}{lV}}$$

Helmholtz resonator

Hermann von Helmholtz
(1821 -1894)

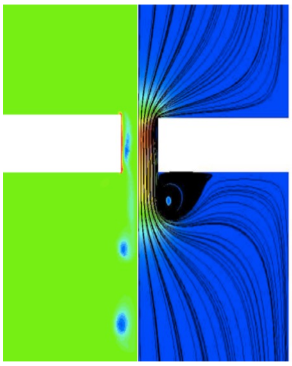


⇔ Mass + Spring resonator

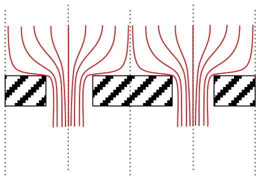
Mass = mass of air in the neck of the resonator

Spring = volume of air in the cavity

Propagation in ducts (1D)



For high level the flow in holes is no longer potential ⇒ creation of vorticity on each half acoustical period.
For very high levels, a shear layer is formed and the vorticity is detached.



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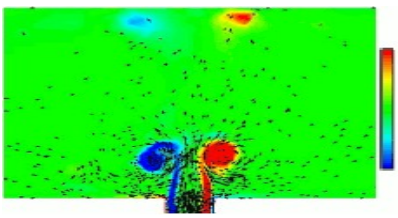
Propagation in ducts (1D)



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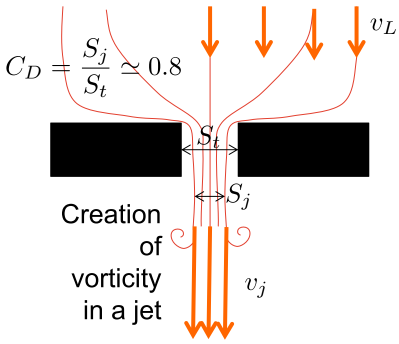
Propagation in ducts (1D)



Compact zone i.e. $\lambda >> \text{hole size}$

Continuity of flow rate $\sigma C_D v_j = v_L$

Bernoulli equation

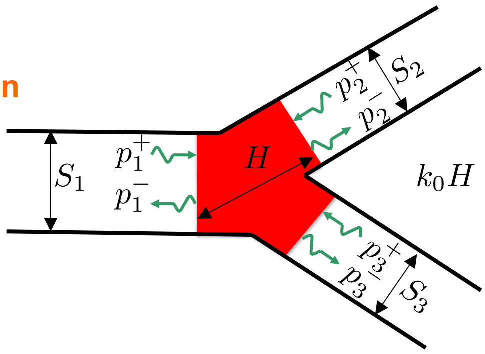
$$p_L + \frac{1}{2}\rho_0 v_L^2 = p_j + \frac{1}{2}\rho_0 v_j^2$$

$$dp = p_L - p_c = \frac{1}{2}\rho_0 \left(\frac{1}{(\sigma C_D)^2} - 1 \right) |v_L| v_L = \frac{1}{2}\rho_0 K |v_L| v_L$$

The impedance is a linear concept. What means "Non Linear impedance ?"

Propagation in ducts (1D)

Bifurcation



Continuity of pressure on node $p_1 = p_2 = p_3$

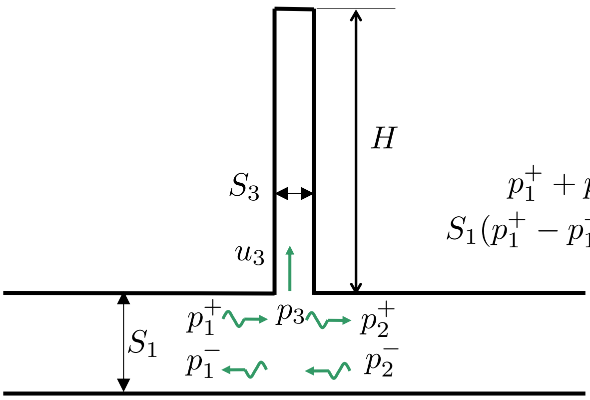
$$p_1^+ + p_1^- = p_2^+ + p_2^- = p_3^+ + p_3^-$$

Continuity of acoustic flow rate $S_1 u_1 + S_2 u_2 + S_3 u_3 = 0$

$$S_1(p_1^+ - p_1^-) + S_2(p_2^+ - p_2^-) + S_3(p_3^+ - p_3^-) = 0$$

Propagation in ducts (1D)

Example: Tube with a closed tube on the side



$$p_1^+ + p_1^+ = p_2^+ = p_3$$
$$S_1(p_1^+ - p_1^+) = S_1 p_2^+ + S_3 Z_c u_3$$

Propagation in ducts (1D)

Example: Tube with a closed tube on the side

$$p_1^+ + p_1^+ = p_2^+ = p_3$$
$$S_1(p_1^+ - p_1^+) = S_1 p_2^+ + S_3 Z_c u_3$$

$$\frac{p_3}{Z_c u_3} = \frac{Z_3}{Z_c} = \frac{1}{\mathrm{j}\tan(k_0 H)}$$

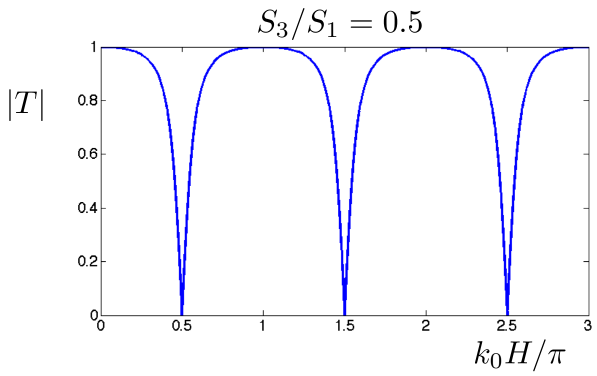
$$p_1^+ + p_1^- = p_2^+$$
$$p_1^+ - p_1^- = p_2^+ \left(1 + \mathrm{j}\frac{S_3}{S_1}\tan(k_0 H)\right)$$

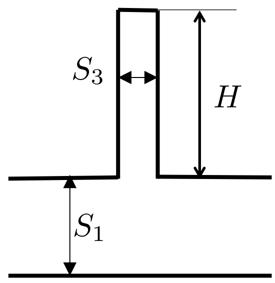
$$T = \frac{p_2^+}{p_1^+} = \frac{2}{2 + \mathrm{j}S_3/S_1 \tan(k_0 H)}$$

Propagation in ducts (1D)

Example: Tube with a closed tube on the side

$$T = \frac{2}{2 + \mathrm{j} S_3/S_1 \tan(k_0 H)}$$





Pass-Band filter

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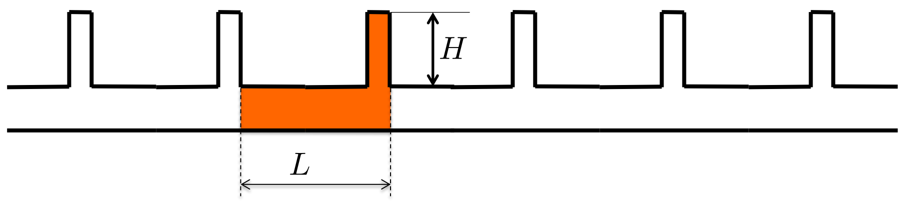
Propagation in ducts (1D)

Periodic structure



periodic cell

Transfer Matrix

$$\begin{pmatrix} p_{n+1}^+ \\ p_{n+1}^- \end{pmatrix} = [T] \begin{pmatrix} p_n^+ \\ p_n^- \end{pmatrix} \Rightarrow \begin{pmatrix} p_N^+ \\ p_N^- \end{pmatrix} = [T]^N \begin{pmatrix} p_0^+ \\ p_0^- \end{pmatrix}$$

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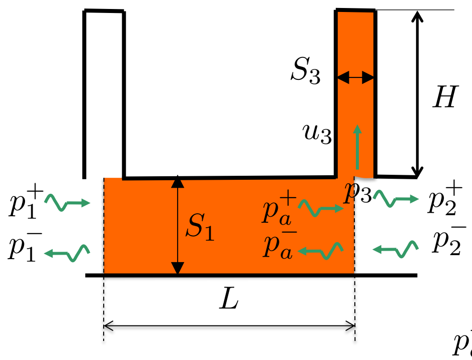
Propagation in ducts (1D)

Periodic structure



$$p_a^+ = p_1^+ \mathrm{e}^{-\mathrm{j}k_0 L}$$
$$p_a^- = p_1^- \mathrm{e}^{\mathrm{j}k_0 L}$$
$$p_a^+ + p_a^+ = p_2^+ + p_2^- = p_3$$
$$S_1(p_a^+ - p_a^+) = S_1(p_2^+ - p_2^-) + S_3 Z_c u_3$$
$$\frac{p_3}{Z_c u_3} = \frac{Z_3}{Z_c} = \frac{1}{\mathrm{j}\tan(k_0 H)}$$

Propagation in ducts (1D)

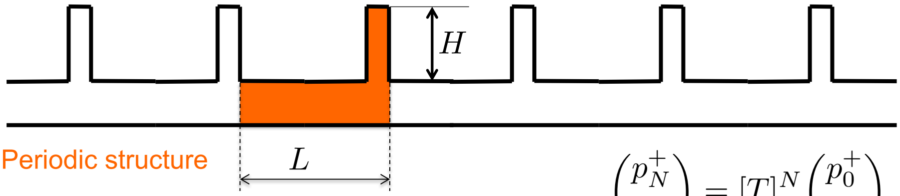
Periodic structure $\qquad Y_3 = \dfrac{S_3 Z_c}{S_1 Z_3}$

$$\begin{pmatrix} 1 & 1 \\ 1+Y_3 & -(1-Y_3) \end{pmatrix} \begin{pmatrix} p_2^+ \\ p_2^- \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{-\mathrm{j}k_0 L} & \mathrm{e}^{\mathrm{j}k_0 L} \\ \mathrm{e}^{-\mathrm{j}k_0 L} & -\mathrm{e}^{\mathrm{j}k_0 L} \end{pmatrix} \begin{pmatrix} p_1^+ \\ p_1^- \end{pmatrix}$$

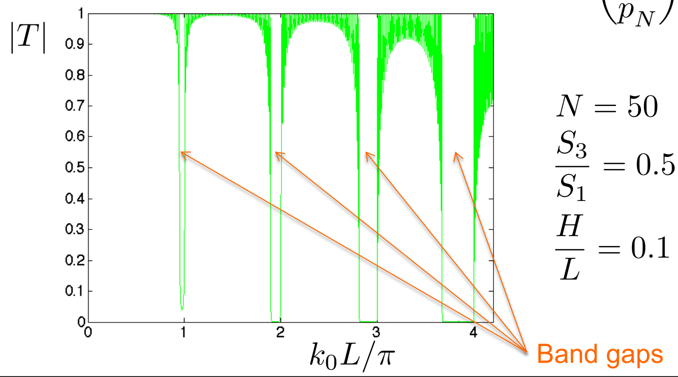
$$\begin{pmatrix} p_2^+ \\ p_2^- \end{pmatrix} = \frac{1}{2} \begin{pmatrix} (1-Y_3)\mathrm{e}^{-\mathrm{j}k_0 L} + \mathrm{e}^{-\mathrm{j}k_0 L} & (1-Y_3)\mathrm{e}^{\mathrm{j}k_0 L} - \mathrm{e}^{\mathrm{j}k_0 L} \\ (1+Y_3)\mathrm{e}^{-\mathrm{j}k_0 L} - \mathrm{e}^{-\mathrm{j}k_0 L} & (1+Y_3)\mathrm{e}^{\mathrm{j}k_0 L} + \mathrm{e}^{\mathrm{j}k_0 L} \end{pmatrix} \begin{pmatrix} p_1^+ \\ p_1^- \end{pmatrix}$$

Transfer Matrix $[T]$

Propagation in ducts (1D)



$$\begin{pmatrix} p_N^+ \\ p_N^- \end{pmatrix} = [T]^N \begin{pmatrix} p_0^+ \\ p_0^- \end{pmatrix}$$



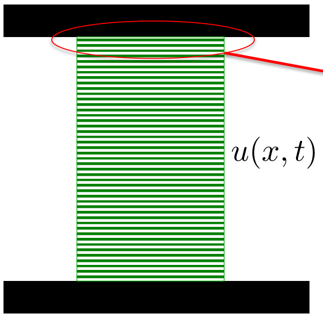
$$N = 50$$

$$\frac{S_3}{S_1} = 0.5$$

$$\frac{H}{L} = 0.1$$

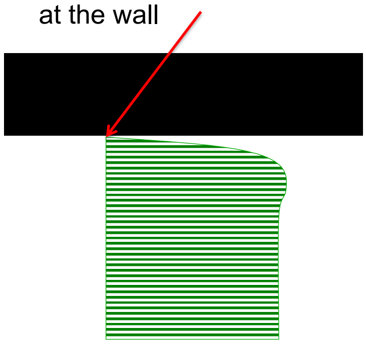
Dissipation in ducts

Dissipation in ducts



Problem with viscosity !

In real life, the velocity had to be 0 at the wall



It is the same for temperature

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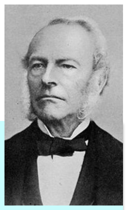
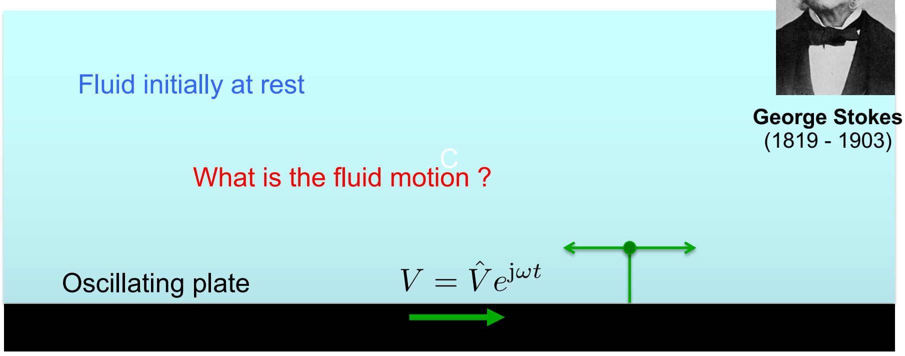
Dissipation in ducts

The second Stokes problem



George Stokes
(1819 - 1903)



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Dissipation in ducts

The second Stokes problem

Incompressible fluid $\vec{\nabla}.\vec{u} = 0$ $\quad \dfrac{\partial u_x}{\partial x} + \dfrac{\partial u_y}{\partial y} = 0$

Navier-Stokes equation along x

$$\frac{\partial u_x}{\partial t} + u_x \cancel{\frac{\partial u_x}{\partial x}} + u_y \cancel{\frac{\partial u_y}{\partial x}} = -\cancel{\frac{1}{\rho_0}\frac{\partial p}{\partial x}} + \nu \left(\cancel{\frac{\partial^2 u_x}{\partial x^2}} + \frac{\partial^2 u_x}{\partial y^2} \right)$$

$\vec{u} = u_x(y,t)\vec{x}$

$$\boxed{\frac{\partial u_x}{\partial t} = \nu \frac{\partial^2 u_x}{\partial y^2}}$$

Diffusion equation

Oscillating plate $\quad V = \hat{V} e^{\mathrm{j}\omega t}$

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Dissipation in ducts

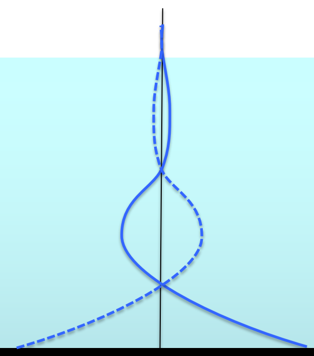
The Stokes problem

$$\frac{\partial^2 \hat{u}_x}{\partial y^2} = \frac{\mathrm{j}\omega}{\nu}\hat{u}_x \qquad \hat{u}_x = A\mathrm{e}^{-\mathrm{j}\beta y} + \cancel{B\mathrm{e}^{\mathrm{j}\beta y}}$$

$$\beta^2 = -\frac{\mathrm{j}\omega}{\nu}$$

$$\beta = (1-\mathrm{j})\sqrt{\frac{\omega}{2\nu}}$$

$$\beta = \frac{1-\mathrm{j}}{\delta_\nu}$$



Oscillating plate $\quad V = \hat{V} e^{\mathrm{j}\omega t}$

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Dissipation in circular ducts

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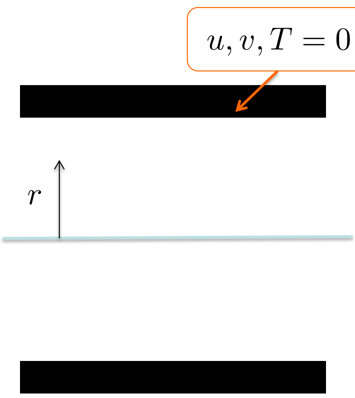
Dissipation in ducts



$$\frac{\partial u}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \nu \Delta_\perp u,$$

$$\frac{\partial p}{\partial r} = 0$$

$$\frac{\partial \rho}{\partial t} + \rho_0 \left(\frac{\partial u}{\partial x} + \frac{1}{r}\frac{\partial\, rv}{\partial r} \right) = 0$$

$$\rho_0 C_p \frac{\partial T}{\partial t} = \frac{\partial p}{\partial t} + \kappa \Delta_\perp T$$

$$\rho = \frac{p}{RT_0} - \rho_0 \frac{T}{T_0}$$

$$\Delta_\perp = \frac{1}{r}\partial_r (r \partial_r)$$

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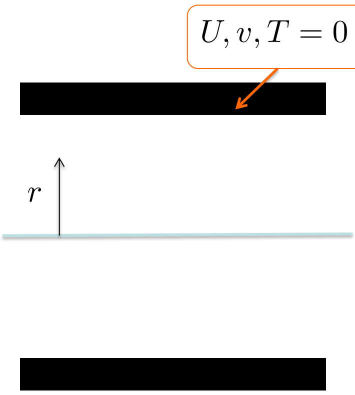
Dissipation in ducts



$$\hat{p}(r) = C$$

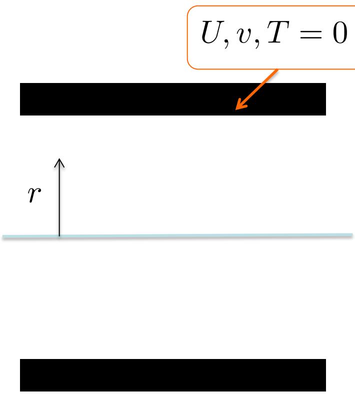
$$\hat{u}(r) = \frac{1}{\rho_0 c_0}\left(AJ_0(\beta r) + KC\right),$$

$$\hat{T}(r) = \frac{1}{\rho_0 C_p}\left(BJ_0(\sigma\beta r) + C\right)$$

$$K = k/(\omega/c_0) \quad \beta^2 = -\frac{\mathrm{j}\omega}{\nu}$$

$$\sigma = \sqrt{P_r}$$

Dissipation in ducts



Applying boundary conditions leads to

$$K^2 = -\frac{\gamma J_0(\beta a)J_0(\sigma\beta a) + (\gamma - 1)J_0(\beta a)J_2(\sigma\beta a)}{J_0(\sigma\beta a)J_2(\beta a)}$$

$$K = k/(\omega/c_0) \quad \beta^2 = -\frac{\mathrm{j}\omega}{\nu} \qquad \sigma = \sqrt{P_r}$$

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Dissipation in ducts

For low shear numbers,

$S_h = a\sqrt{\omega/\nu} \ll 1$

Laminar isothermal flow

For high shear numbers,

$S_h = a\sqrt{\omega/\nu} \gg 1$

$$K = 1 + (1-i)\alpha_0$$

$$\alpha_0 = \frac{1}{\sqrt{2}\,S_h}\left(1 + \frac{\gamma - 1}{\sigma}\right)$$

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Porous material with rigid frame

- Use of the multi-scale asymptotic method

⇒ 2 separated problems to solve

Viscous incompressible flow in the microstructure

Leads to the calculation of the effective density $\rho_e(\omega)$

$$\mathrm{j}\omega\rho_e(\omega)\mathbf{v} = -\nabla p$$

for an isotropic material

$$\mathrm{j}\omega\rho_{eij}(\omega)v_j = -\frac{\partial p}{\partial x_i}$$

in the general case

Thermal diffusion in the microstructure

Leads to the calculation of the effective compressibility $\kappa_e(\omega)$

$$\mathrm{j}\omega\kappa_e^{-1}(\omega)p = -\nabla.\mathbf{v}$$

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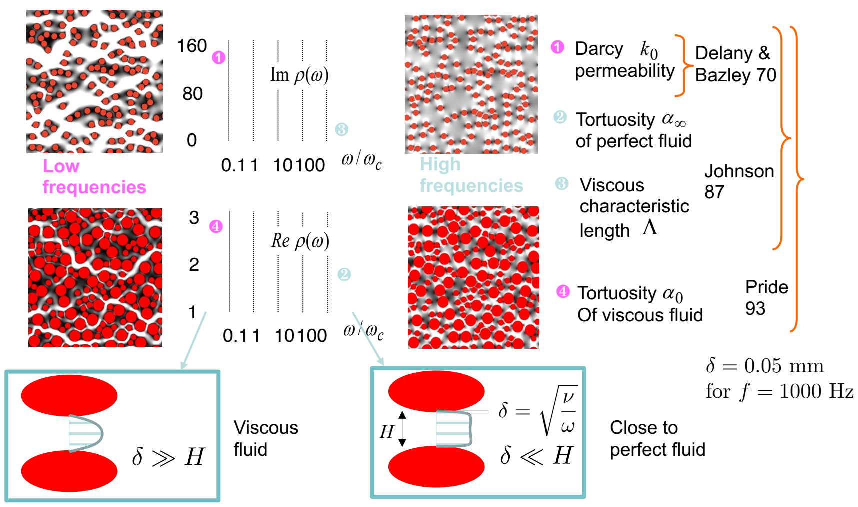
Porous material with rigid frame

$$\rho_0 \longrightarrow \rho_{e_{ij}}(\omega) = \rho_0 \alpha_{ij}(\omega)$$ Dynamic tortuosity



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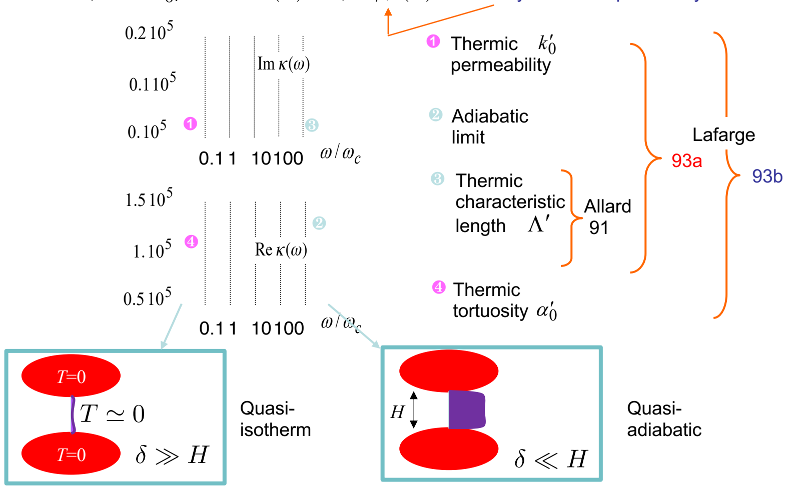
Porous material with rigid frame

$$\kappa_a = \gamma P_0 = c_0^2 \rho_0 \longrightarrow \kappa_e(\omega) = \gamma P_0 / \beta(\omega)$$ Dynamic compressibility



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Porous material with rigid frame

Model with six parameters:
the porosity Φ
the tortuosity α_∞
the viscous and thermal permeabilities k_0 and k'_0
the viscous and thermal characteristic lengths Λ and Λ'

In practice
Resistivity $\sigma = \rho_0 \nu / k_0$
k'_0 can be approximated by $k'_0 = \Phi \Lambda'^2 / 8$

⇒ 5 independent parameters

Effective density
$$\rho_e = \frac{\rho_0 \alpha_\infty}{\Phi}\left(1 + \frac{1}{jx}\left[1 + \frac{M}{2} jx\right]^{1/2}\right)$$

Viscous reduced frequency $x = \dfrac{\omega \alpha_\infty k_0}{\nu \Phi}$ Viscous shape factor $M = \dfrac{8 k_0 \alpha_\infty}{\Phi \Lambda^2}$

Effective compressibility
$$\kappa_e = \frac{\gamma_0 P_0}{\Phi}\left(\gamma - (\gamma - 1)\left(1 + \frac{1}{jx'}\left[1 + \frac{M'}{2} jx'\right]^{1/2}\right)^{-1}\right)^{-1}$$

Thermal reduced frequency $x' = \dfrac{\omega P_r k'_0}{\nu \Phi}$ Thermal shape factor $M' = \dfrac{8 k'_0}{\Phi \Lambda'^2}$

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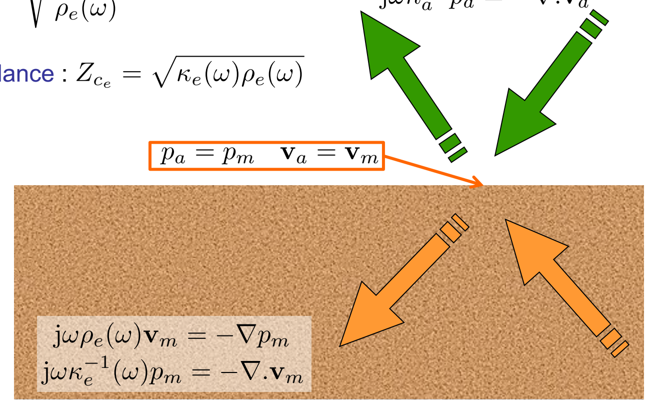
Porous material with rigid frame

From $\rho_e(\omega)$ and $\kappa_e(\omega)$:

Phase Velocity : $c_e = \sqrt{\dfrac{\kappa_e(\omega)}{\rho_e(\omega)}}$

Characteristic Impedance : $Z_{c_e} = \sqrt{\kappa_e(\omega)\rho_e(\omega)}$

$$\mathrm{j}\omega\rho_0 \mathbf{v}_a = -\nabla p_a$$
$$\mathrm{j}\omega\kappa_a^{-1} p_a = -\nabla . \mathbf{v}_a$$



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