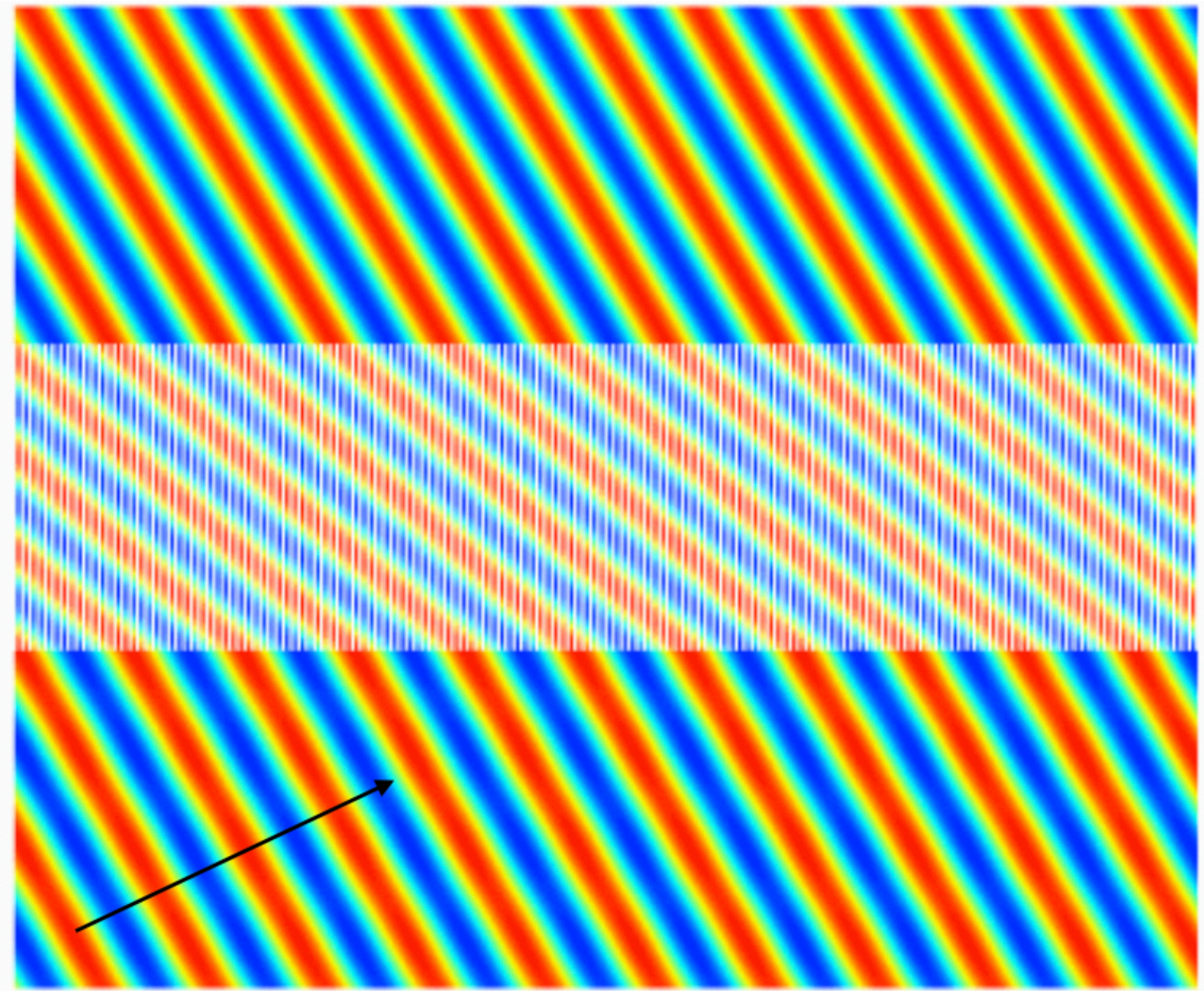
Homogenization

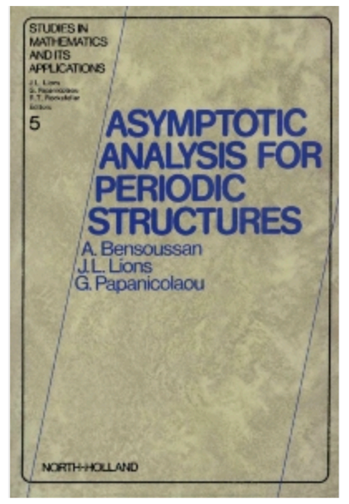


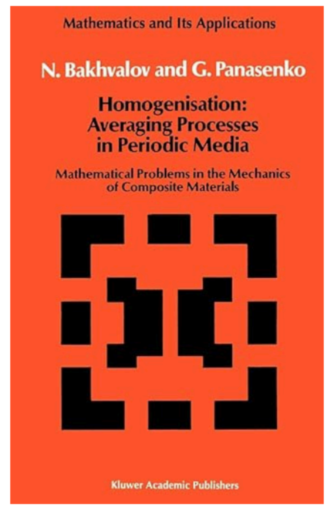
Homogenization

Homogenization aims to replace a micro-structured (inhomogeneous) medium with an equivalent, homogeneous, one

This technique is based on an asymptotic analysis with two scales (spatial in its classical form) defining a small parameter

By its very nature, homogenization is well-suited for modeling structures that rely on metamaterials.



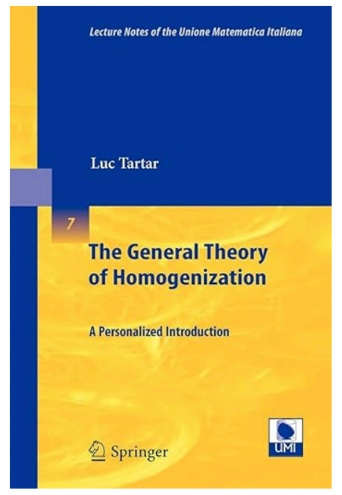


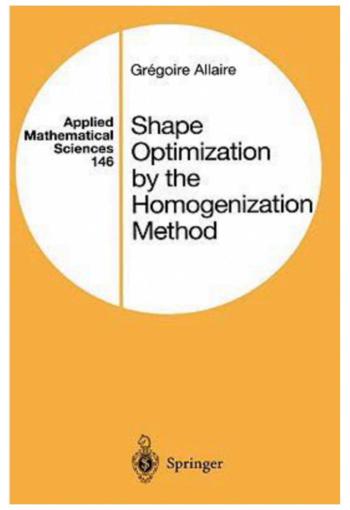
Alain Bensoussan, Jacques-Louis Lions, Georges Papanicolaou (1978)
Asymptotic methods in periodic media

Nikolai Bakhvalov and Grigori Panasenko (1989)
Homogenisation: Averaging Processes in Periodic Media: Mathematical Problems in the Mechanics of Composite Materials

Grégoire Allaire (2002),
Shape optimization by the homogenization method

Luc Tartar (2009),
The General Theory of Homogenization, A Personalized Introduction,



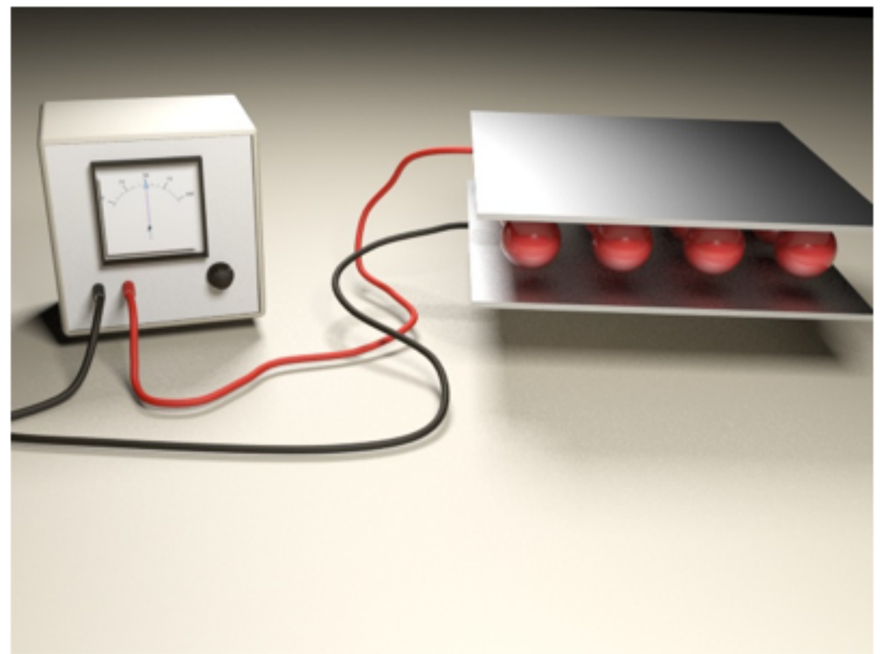


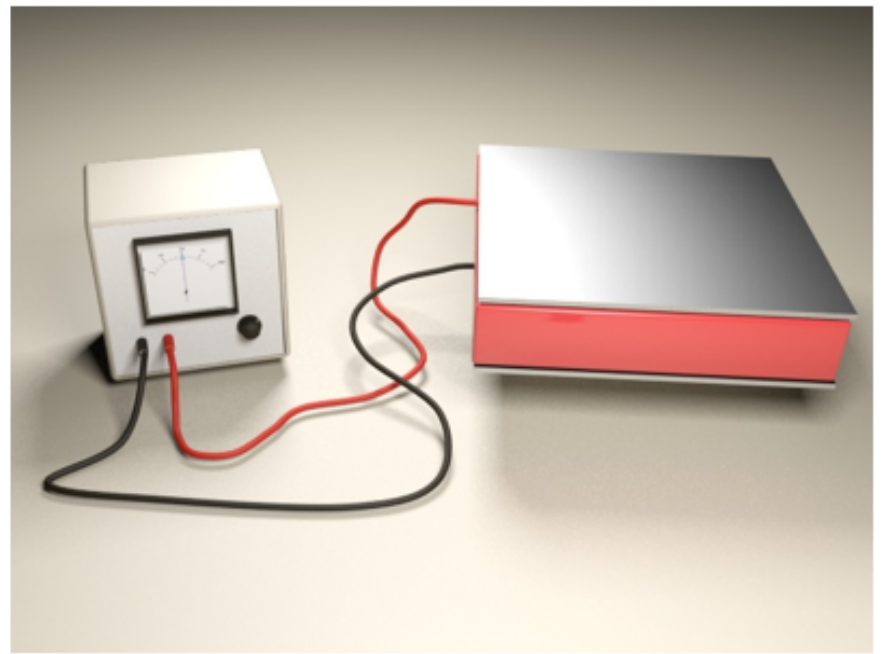
This technique arrives recently in the context of waves

- to deal with locally resonant structures/metamaterials
- to deal with metasurfaces/metainterface

Homogenization

- Homogenization aims to replace a micro-structured (inhomogeneous) medium with an equivalent, homogeneous, one



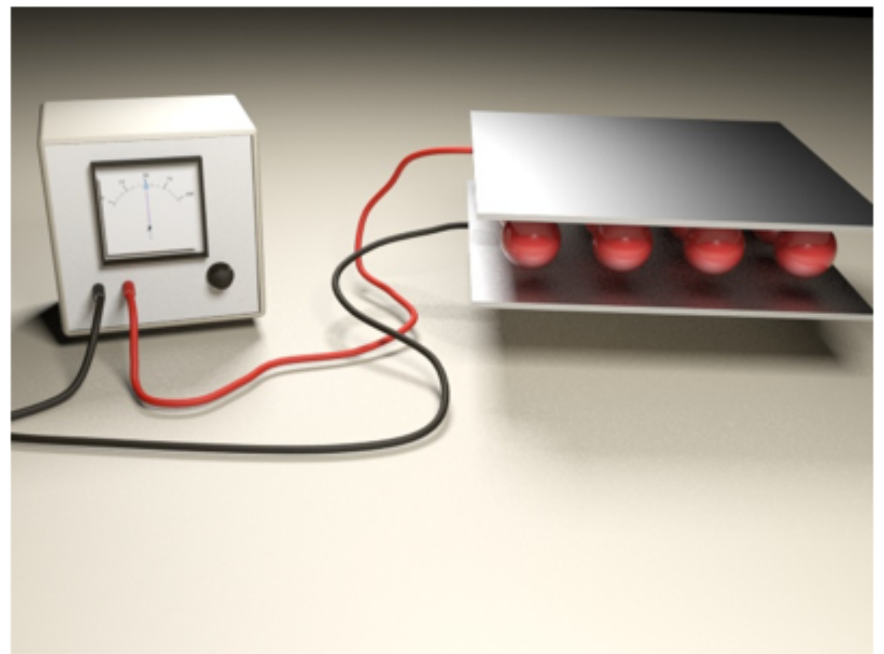


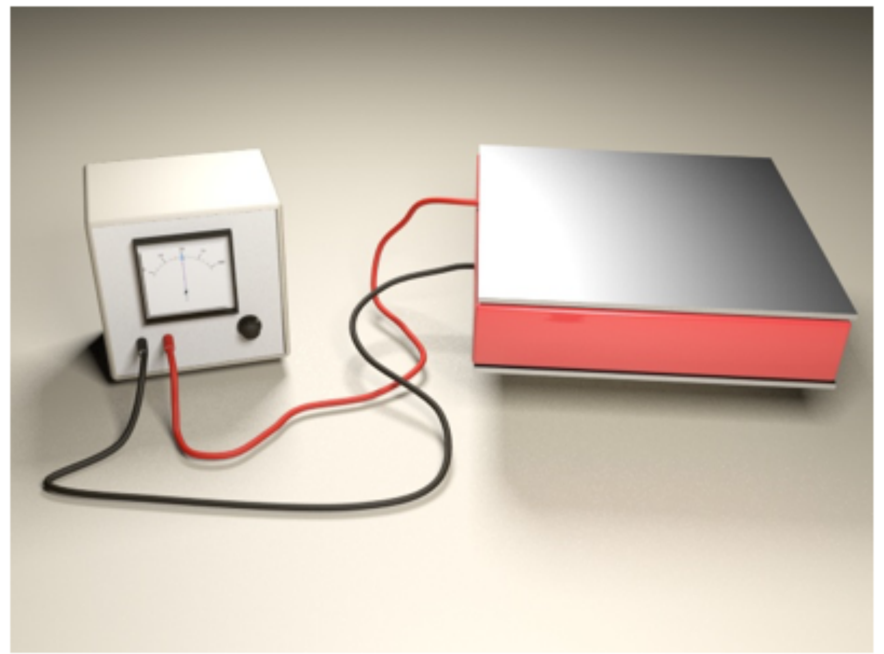
same response (say voltage)

http://people.ee.duke.edu/~drsmith/metamaterials/metamaterials_homogenization.htm

Homogenization

- Homogenization aims to replace a micro-structured (inhomogeneous) medium with an equivalent, homogeneous, one





same response (say voltage)

http://people.ee.duke.edu/~drsmith/metamaterials/metamaterials_homogenization.htm

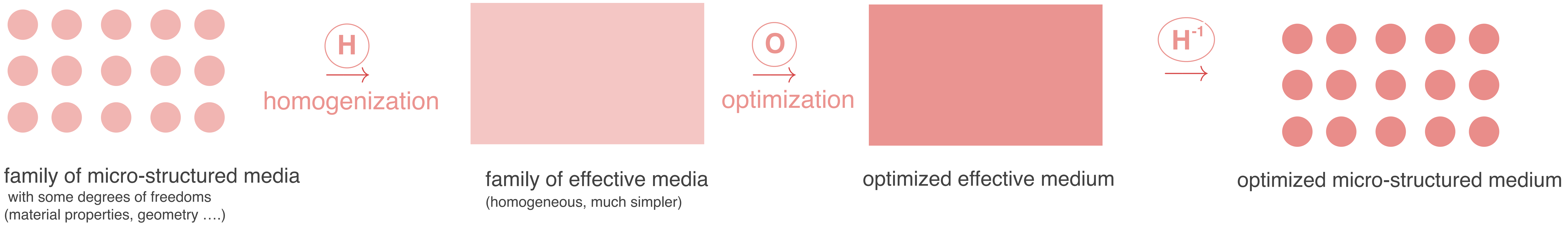
- Why doing so ?

because the effective model can provide a physical picture of the observed phenomena (e.g. the conductivity or refraction index).

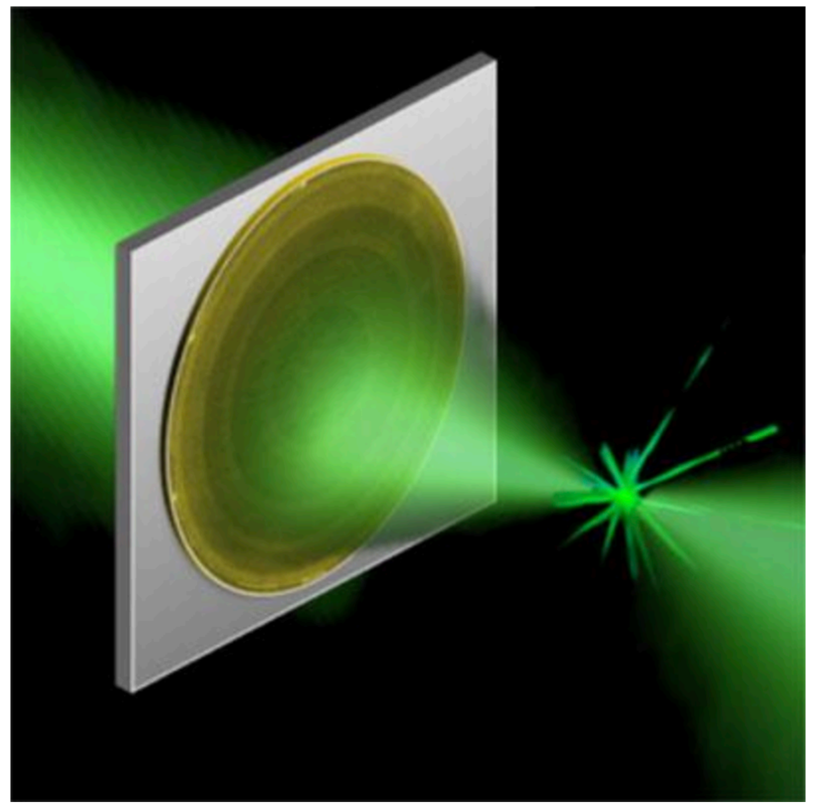
because it is much simpler theoretically to optimise the properties of a homogeneous medium than those of an inhomogeneous medium.

Homogenization

Homogenization and optimization

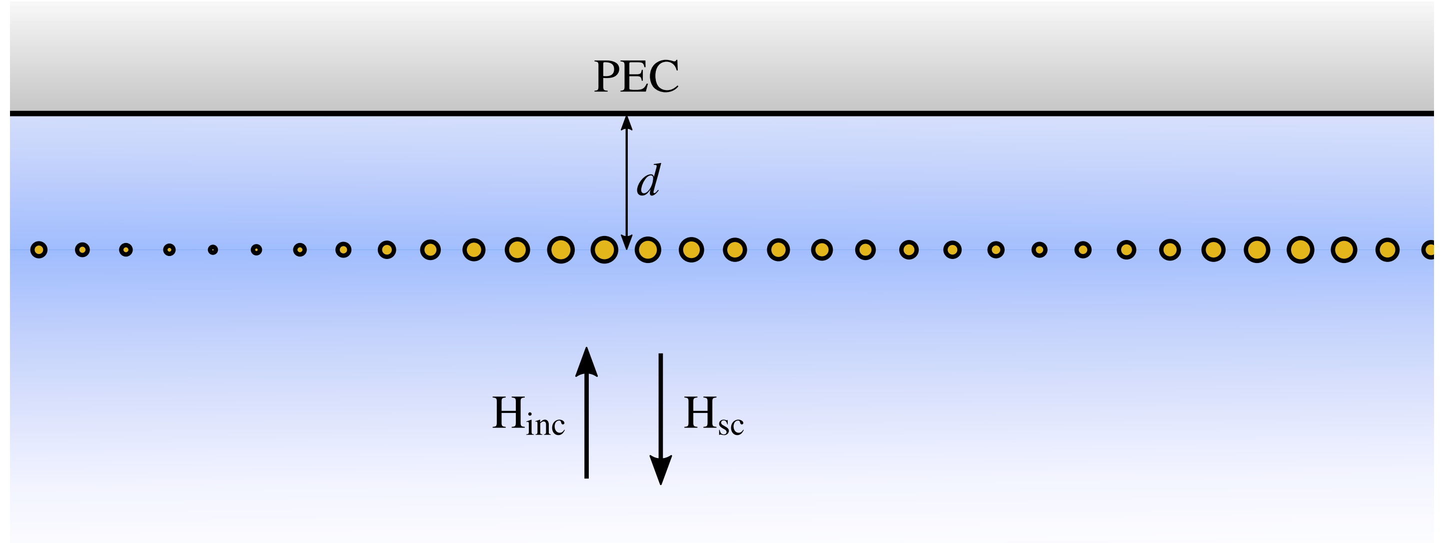


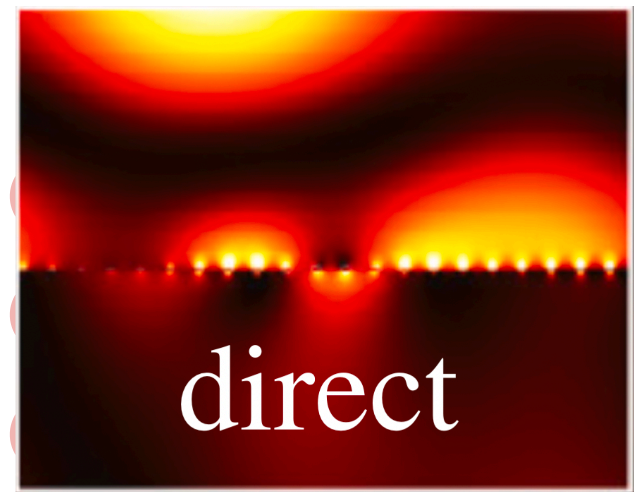
Homogenization offers a one-to-one correspondance between the actual medium (difficult to deal with) and an equivalent one (easier to deal with).



Homogenization

and optimization





family of micro-structured media
with some degrees of freedoms
(material properties, geometry ….)

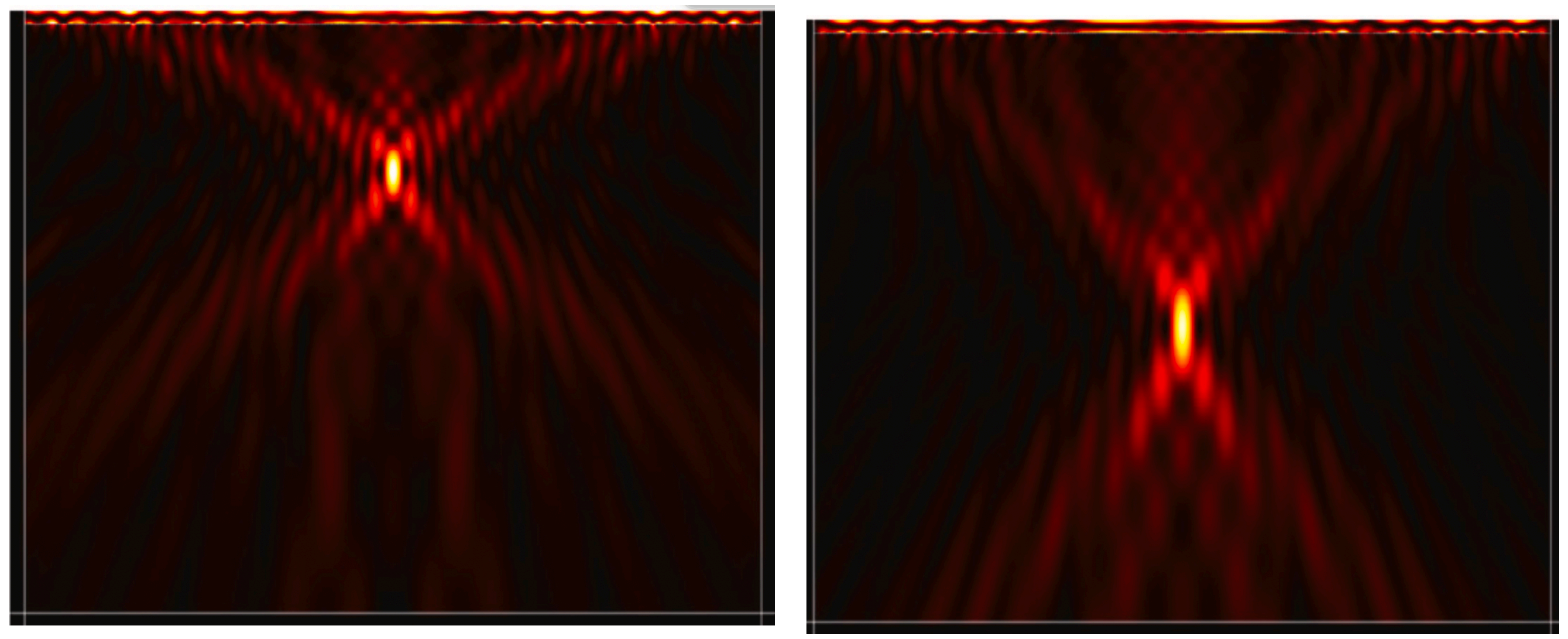
H

homogenization

family of effective media
(homogeneous, much simpler)

O H^{-1}

optimization



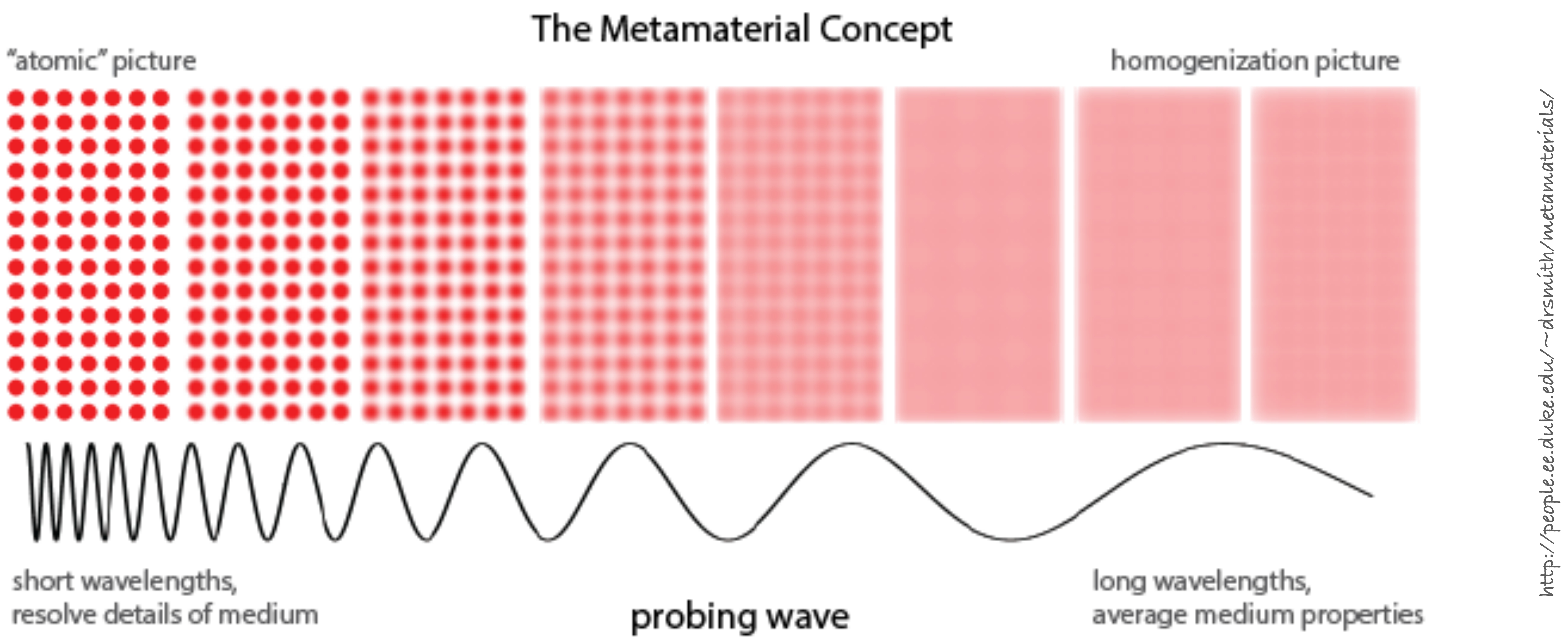
Homogenization offers a one-to-one correspondance between the actual medium (difficult to deal with) and an equivalent one (easier to deal with).

Optimization of plasmonic metasurfaces: a homogenization-based design
N Lebbe, K Pham, A Maurel - Journal of Computational Physics, 2023

Homogenization

- Can we homogenise any structure ?   NO

In its classical form, homogenization makes sense if the wavelength is (much) larger than the spacing (**subwavelength micro-structures**) and it concerns **periodic micro-structures**



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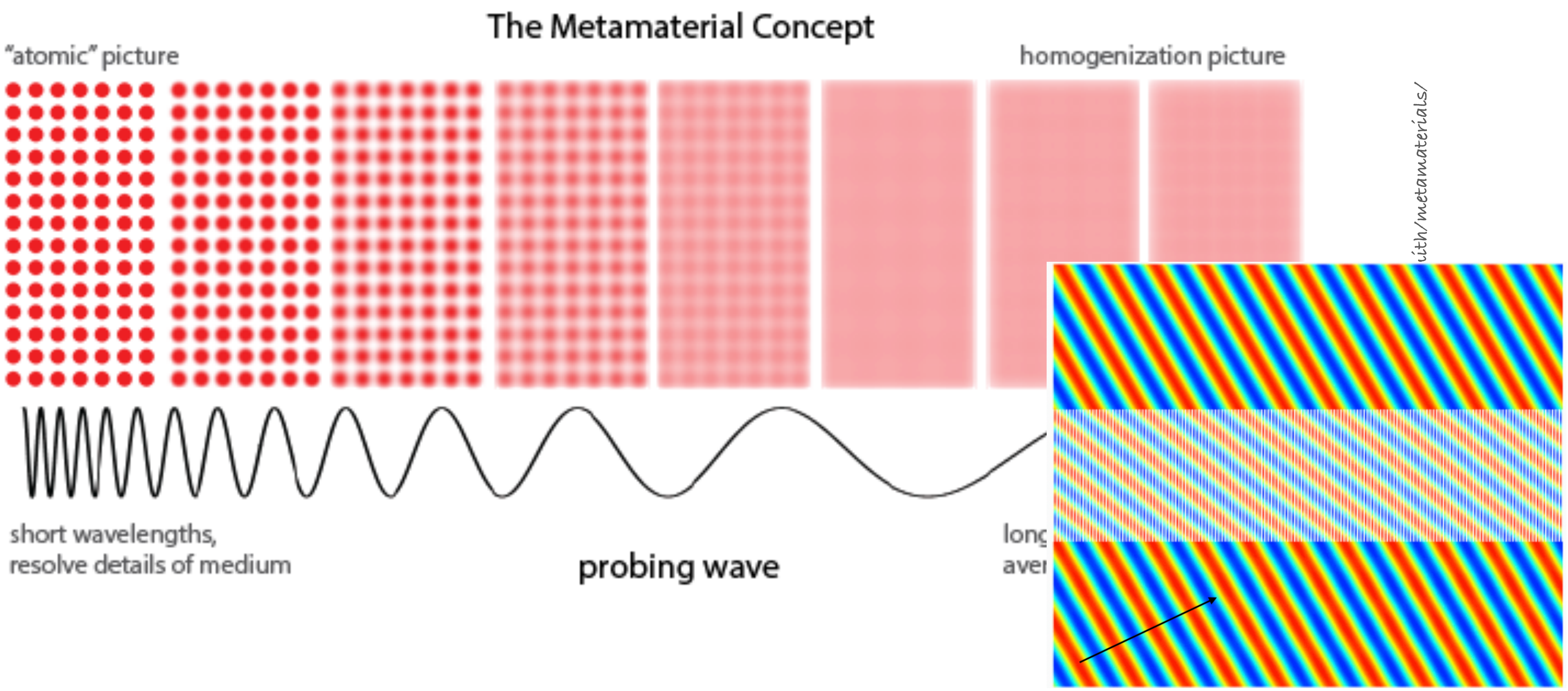
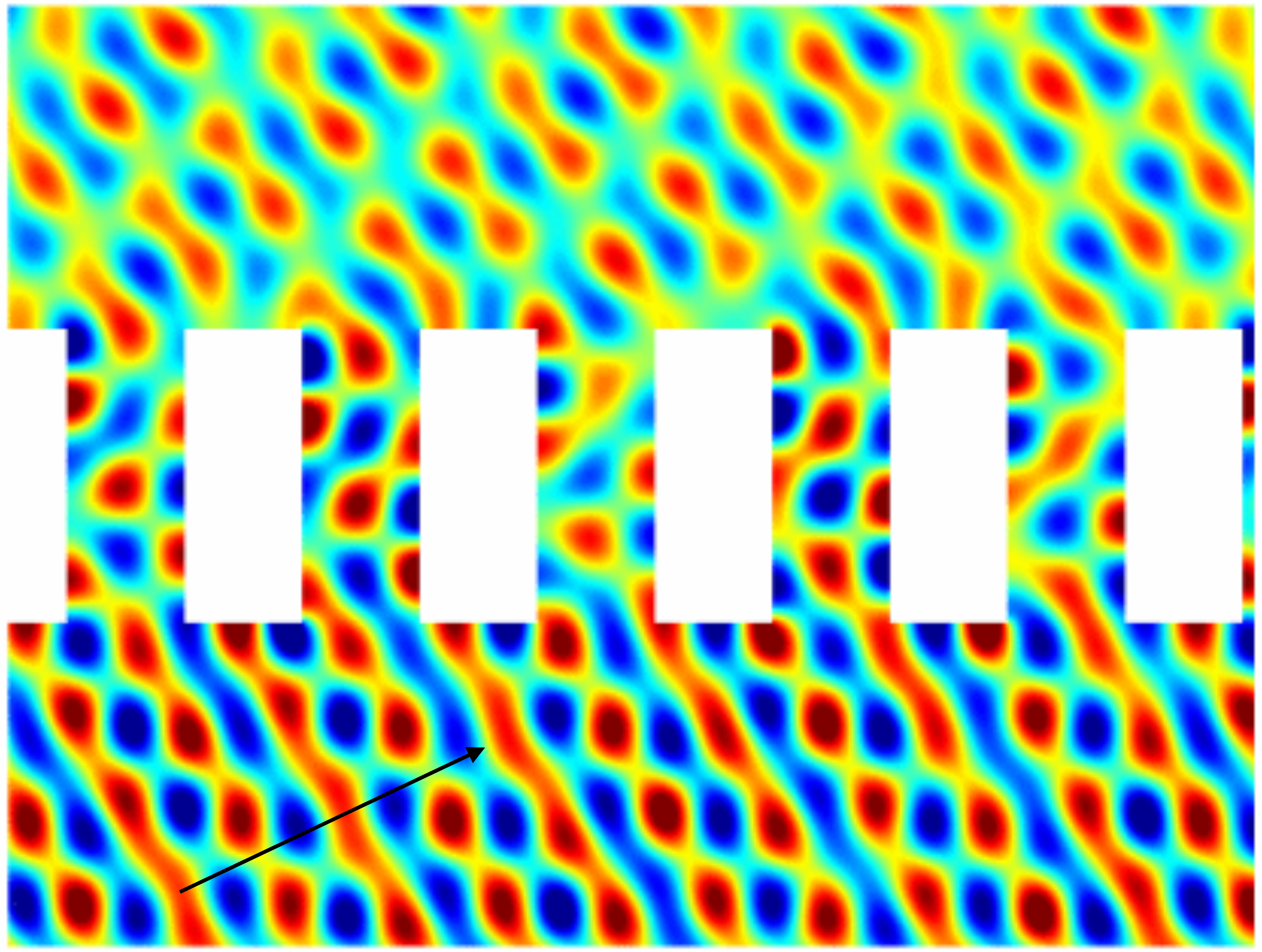


Homogenization

- Can we homogenise any structure ? NO

FROM THE ATOMIC PICTURE TO THE HOMOGENIZATION PICTURE

$kh = \mathbf{16}$

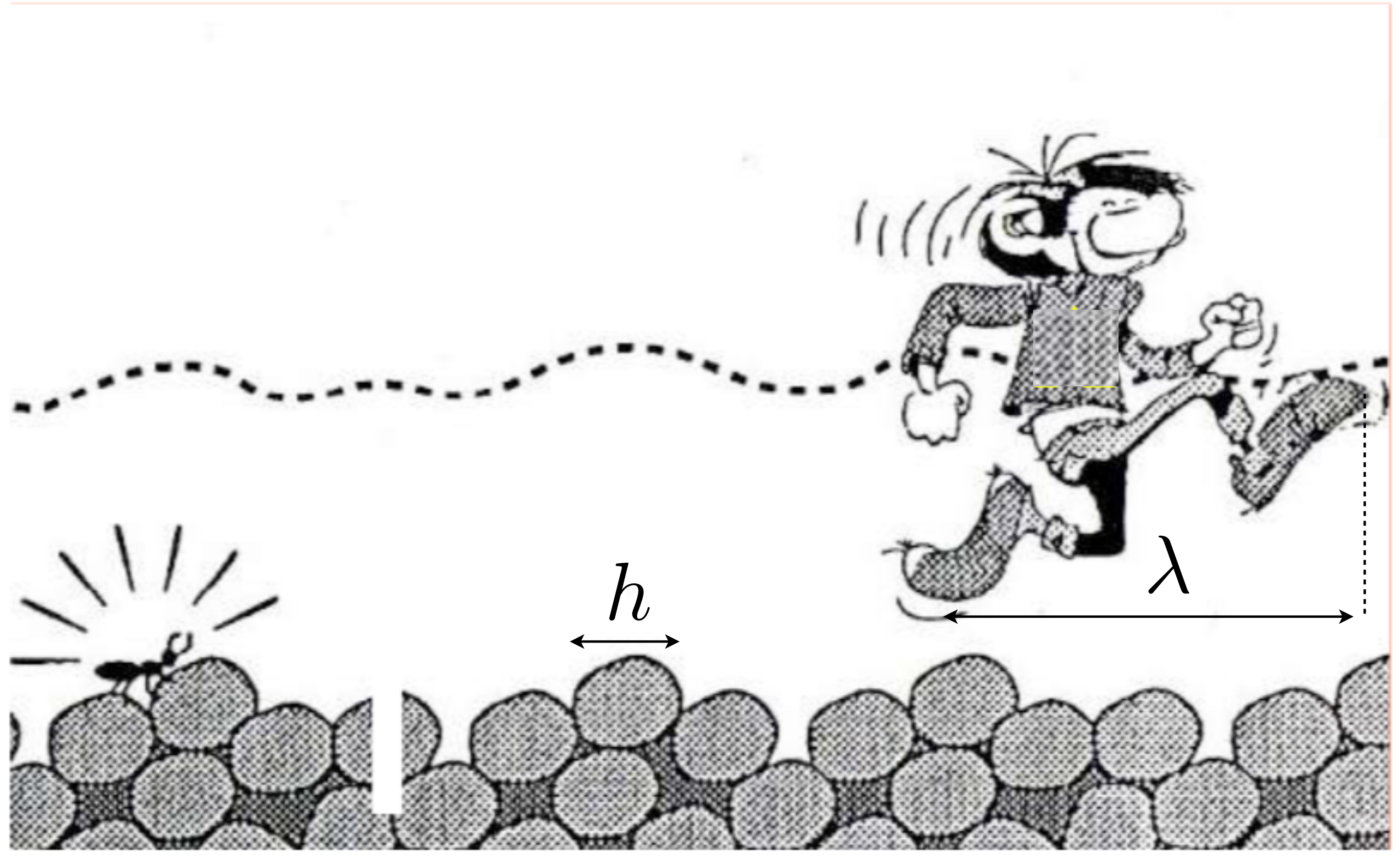


$k = 2\pi/\lambda$

Homogenization

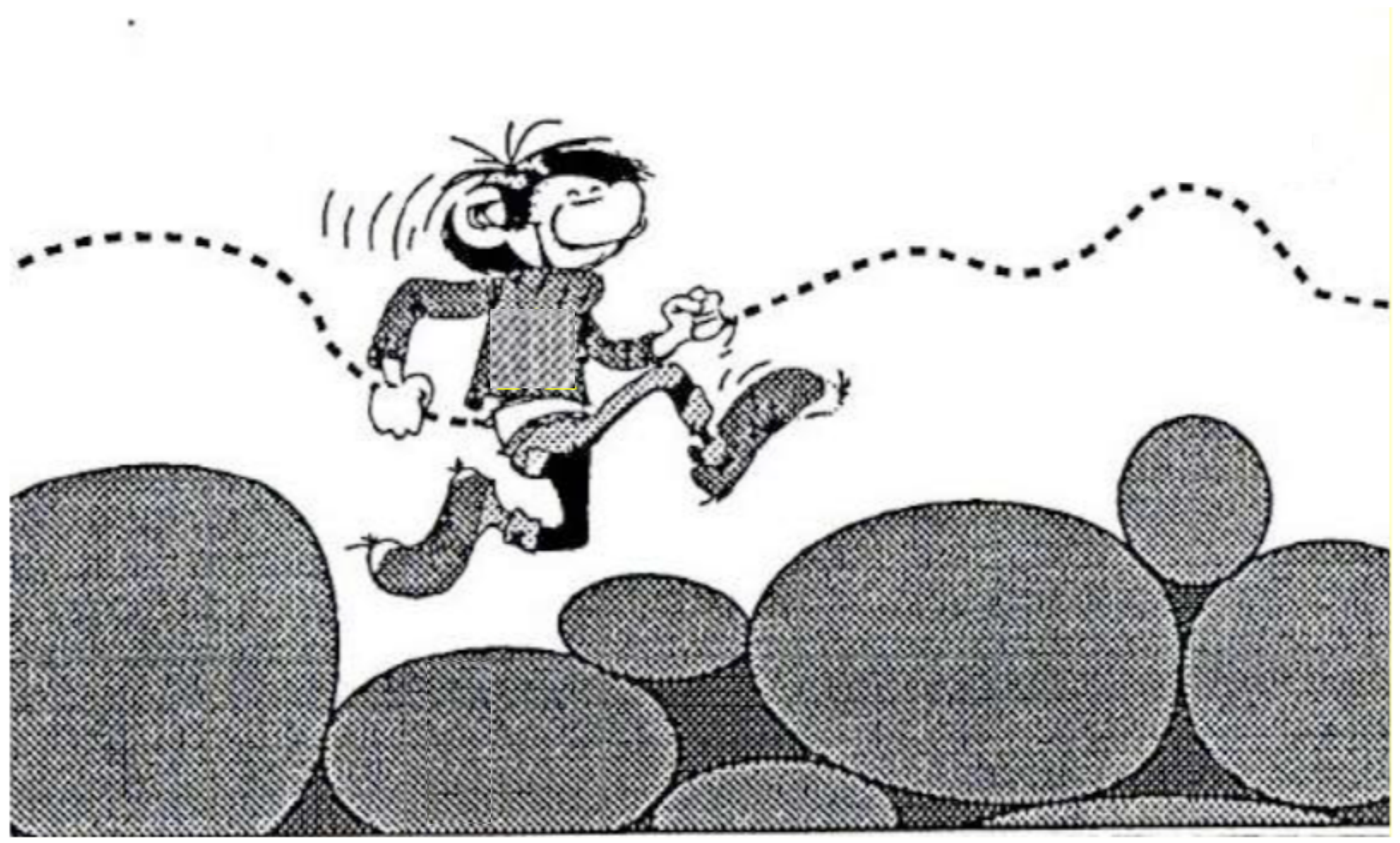
- Can we homogenise any structure ? NO

$h \ll \lambda$



Effective speed

$h \sim \lambda$

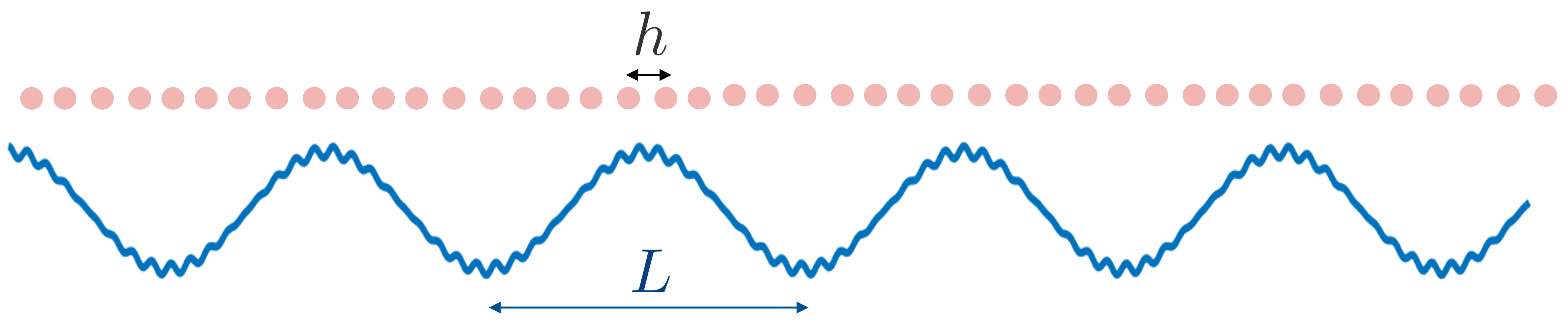


[Boutin et Auriault]

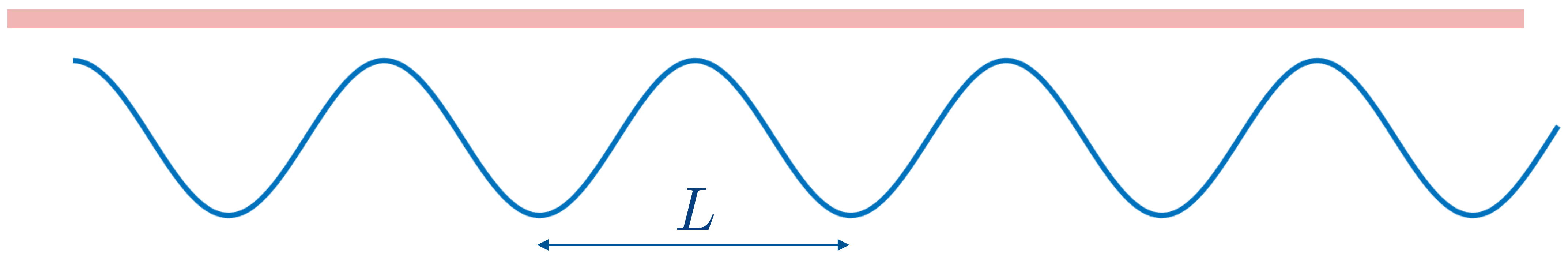
Homogenization

« separation of scales »
homogenization aims to capture the effect of the small scale in an « averaged sense »

the actual problem



the homogenized problem



means some averaging process during homogenization

Homogenization

The classical (non resonant) two-scale homogenization

Homogenization

We shall work with the simplest wave equation :

wave equation written in terms of a scalar field $p(\mathbf{x},t)$ a vector field $\mathbf{u}(\mathbf{x},t)$
with material parameters $a(\mathbf{x})$ and $b(\mathbf{x})$

$$\frac{\partial \mathbf{u}}{\partial t} = -a\,\nabla p \qquad \mathrm{div}\mathbf{u} + b\,\frac{\partial p}{\partial t} = 0 \qquad \longrightarrow \qquad \frac{\partial^2 p}{\partial t^2} - \frac{a}{b}\,\Delta p = \mathbf{0}$$

continuity of p and of $\mathbf{u}\cdot\mathbf{n}$ at the interfaces

This wave equation applies in many contexts of waves

	scalar field *p*	vector field ***u***	material parameter *a*	material parameter *b*
acoustics (full 3d)	pressure *p*	velocity ***u***	inverse of mass density ρ	inverse of bulk modulus $B=\rho c^2$
electromagnetism (2d polarized)	out-of-plane magnetic field *H*	auxiliary field linked to the in-plane electric field ***E***	inverse of permittivity ε	permeability μ
elastodynamics (2d)	out-of-plane velocity *u*	in-plane vector stress σ	shear modulus μ	mass density ρ

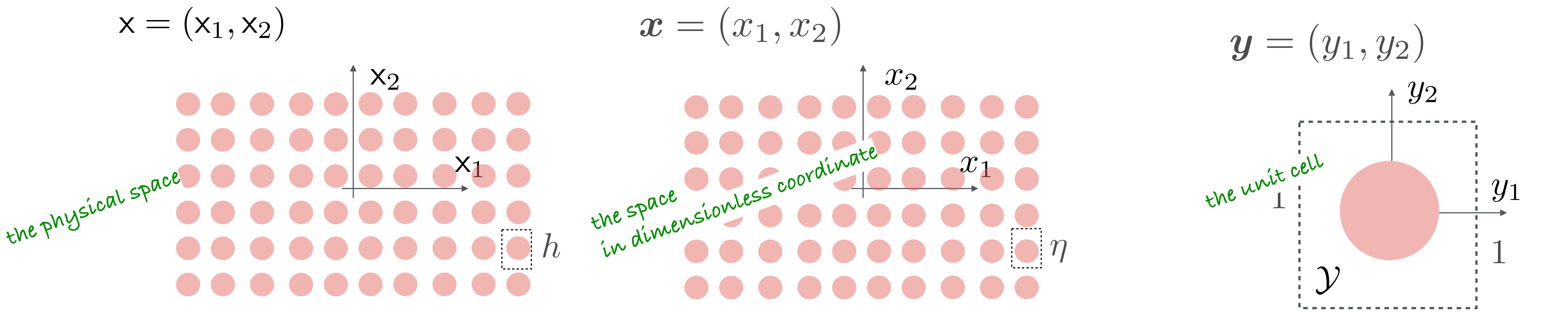
Homogenization

$$\frac{\partial \mathbf{u}}{\partial t} = -a\, \nabla p, \qquad \mathrm{div}\mathbf{u} + b\frac{\partial p}{\partial t} = 0, \qquad p \text{ and } \boldsymbol{u}\cdot\mathbf{n} \text{ continuous}$$

The separation of the scales allows to define the small parameter $\eta = kh \ll 1$

$k = \frac{1}{L}$ is the typical wavenumber

Working spaces



$\boldsymbol{x} = k\mathrm{x}$

$\boldsymbol{y} = \frac{\mathrm{x}}{h}$, hence $\boldsymbol{y} = \frac{\boldsymbol{x}}{\eta}$

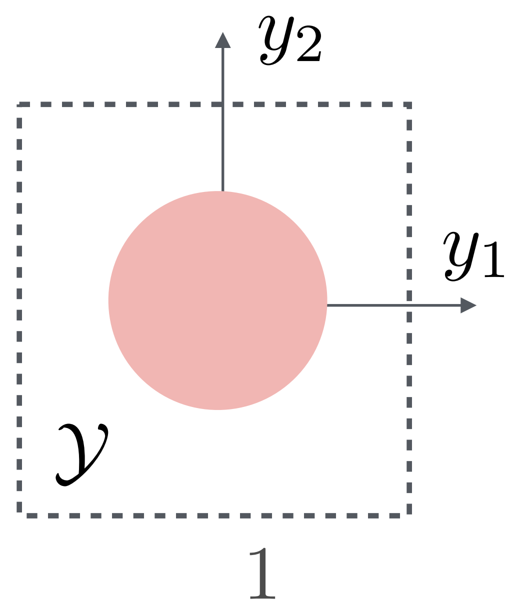
Homogenization

$$\frac{\partial \mathbf{u}}{\partial t} = -a\,\nabla p, \qquad \mathrm{div}\mathbf{u} + b\frac{\partial p}{\partial t} = 0, \qquad p \text{ and } \boldsymbol{u}\cdot\mathbf{n} \text{ continuous}$$

Two-scale expansions

$$p = \sum_{n=0}^{\infty} \eta^n p^n(\boldsymbol{x},\boldsymbol{y},t), \quad \boldsymbol{u} = \sum_{n=0}^{\infty} \eta^n \boldsymbol{u}^n(\boldsymbol{x},\boldsymbol{y},t), \qquad p^n, \boldsymbol{u}^n\cdot\boldsymbol{n} \text{ continuous}, \ p^n, \ \boldsymbol{u}^n \text{ periodic in } \mathcal{Y}$$

$$a(\boldsymbol{y}), b(\boldsymbol{y}). \quad \nabla \longrightarrow \nabla_{\boldsymbol{x}} + \frac{1}{\eta}\nabla_{\boldsymbol{y}}$$



We want to determine the equations satisfied by the $P^n(\boldsymbol{x},t)$ and the $\boldsymbol{U}^n(\boldsymbol{x},t)$ defined as

$$P^n(\boldsymbol{x},t) = \int_{\mathcal{Y}} p^n(\boldsymbol{x},\boldsymbol{y},t)\mathrm{d}\boldsymbol{y}, \quad \boldsymbol{U}^n(\boldsymbol{x},t) = \int_{\mathcal{Y}} \boldsymbol{u}^n(\boldsymbol{x},\boldsymbol{y},t)\mathrm{d}\boldsymbol{y}$$

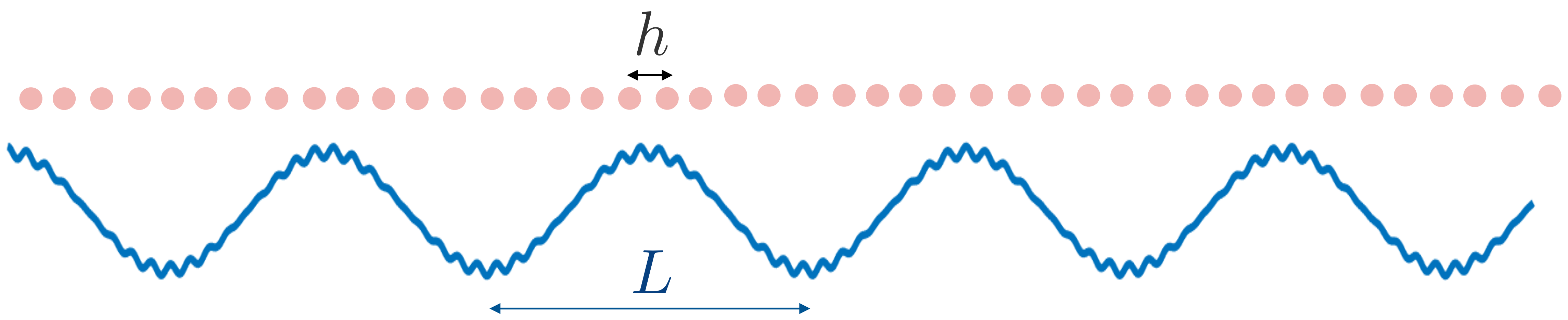
Homogenization

parenthesis on the meaning of these macroscopic fields

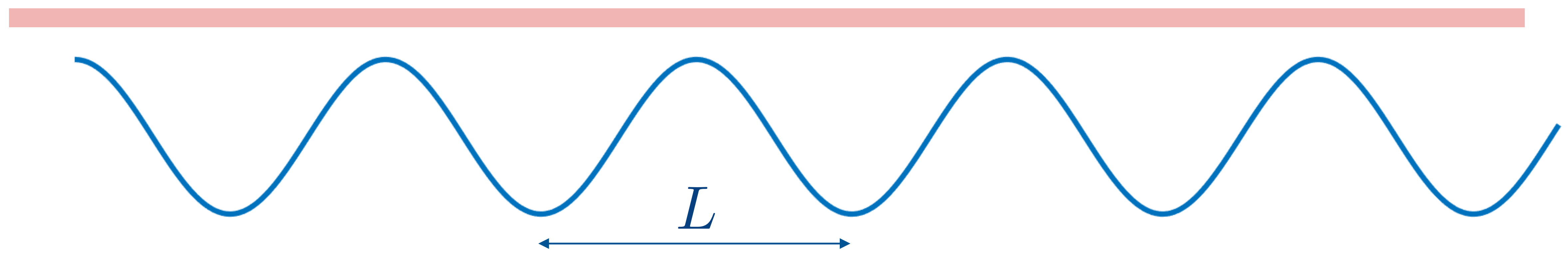
We want to determine the equations satisfied by the $P^n(\boldsymbol{x},t)$ and the $\boldsymbol{U}^n(\boldsymbol{x},t)$ defined as

$$P^n(\boldsymbol{x},t) = \int_{\mathcal{Y}} p^n(\boldsymbol{x},\boldsymbol{y},t)\mathrm{d}\boldsymbol{y}, \quad \boldsymbol{U}^n(\boldsymbol{x},t) = \int_{\mathcal{Y}} \boldsymbol{u}^n(\boldsymbol{x},\boldsymbol{y},t)\mathrm{d}\boldsymbol{y}$$

$$p = \sum_{n=0}^{\infty} \eta^n p^n(\boldsymbol{x},\boldsymbol{y},t),$$



$$P = \sum_{n=0}^{\infty} \eta^n P^n(\boldsymbol{x},t),$$



means some averaging process during homogenization

Homogenization

$$\frac{\partial \mathbf{u}}{\partial t} = -a\,\nabla p, \qquad \operatorname{div}\mathbf{u} + b\frac{\partial p}{\partial t} = 0,$$

$$\nabla \longrightarrow \nabla_{\boldsymbol{x}} + \frac{1}{\eta}\nabla_{\boldsymbol{y}}$$

$\nabla_{\boldsymbol{y}} p^0 = \mathbf{0}$	$\operatorname{div}_{\boldsymbol{y}} \boldsymbol{u}^0 = 0,$ $\frac{\partial}{\partial t}\boldsymbol{u}^0 = -a(\boldsymbol{y})\left(\nabla_{\boldsymbol{x}} p^0 + \nabla_{\boldsymbol{y}} p^1\right)$	$\operatorname{div}_{\boldsymbol{x}} \boldsymbol{u}^0 + \operatorname{div}_{\boldsymbol{y}} \boldsymbol{u}^1 + b(\boldsymbol{y})\frac{\partial p^0}{\partial t} = 0.$

$$p^0(\boldsymbol{x}, \boldsymbol{y}, t) = P^0(\boldsymbol{x}, t)$$

$$\operatorname{div}_{\boldsymbol{x}} \boldsymbol{U}^0 + b_{\text{eff}}\frac{\partial P^0}{\partial t} = 0, \quad b_{\text{eff}} = \int_{\mathcal{Y}} b(\boldsymbol{y})\mathrm{d}\boldsymbol{y}.$$

this problem is set in $\mathcal{Y}$ on $(\boldsymbol{u}^0, p^1)$

It has to be complemented by boundary conditions

$p^1, \boldsymbol{u}^0 \cdot \boldsymbol{n}$ continuous, $p^1,\ \boldsymbol{u}^0$ periodic in $\mathcal{Y}$

Homogenization

$$\frac{\partial \mathbf{u}}{\partial t} = -a\,\nabla p, \qquad \mathrm{div}\mathbf{u} + b\frac{\partial p}{\partial t} = 0,$$

$$\nabla \longrightarrow \nabla_{\boldsymbol{x}} + \frac{1}{\eta}\nabla_{\boldsymbol{y}}$$

$\nabla_{\boldsymbol{y}} p^0 = \mathbf{0}$	$\mathrm{div}_{\boldsymbol{y}} \boldsymbol{u}^0 = 0,$ $\frac{\partial}{\partial t}\boldsymbol{u}^0 = -a(\boldsymbol{y})\left(\nabla_{\boldsymbol{x}} p^0 + \nabla_{\boldsymbol{y}} p^1\right)$	$\mathrm{div}_{\boldsymbol{x}} \boldsymbol{u}^0 + \mathrm{div}_{\boldsymbol{y}} \boldsymbol{u}^1 + b(\boldsymbol{y})\frac{\partial p^0}{\partial t} = 0.$

$$\frac{\partial \boldsymbol{u}^0}{\partial t} = -\frac{\partial P^0}{\partial x_i}(\boldsymbol{x}, t) a\left(\boldsymbol{e}_i + \nabla_{\boldsymbol{y}} Q_i\right),$$

$$\frac{\partial U_j^0}{\partial t}(\boldsymbol{x}, t) = -a_{ji}\frac{\partial P^0}{\partial x_i}(\boldsymbol{x}, t), \qquad a_{ji} = \int_{\mathcal{Y}} \left[a\left(\delta_{ij} + \frac{\partial Q_i}{\partial y_j}\right)\right] \mathrm{d}\boldsymbol{y},$$

$$\mathrm{div}_{\boldsymbol{y}}\left(a\left(\boldsymbol{e}_i + \nabla_{\boldsymbol{y}} Q_i\right)\right) = 0,$$

$$Q_i, a(\boldsymbol{e}_i + \nabla_{\boldsymbol{y}} Q_i)\cdot \boldsymbol{n} \text{ continuous},$$

$$Q_i, a(\boldsymbol{e}_i + \nabla_{\boldsymbol{y}} Q_i) \text{ periodic in } \mathcal{Y}$$

Homogenization

wave equation written in terms of a scalar field $p(\mathbf{x},t)$ a vector field $\mathbf{u}(\mathbf{x},t)$
with material parameters $a(\mathbf{x})$ and $b(\mathbf{x})$

$$\frac{\partial \mathbf{u}}{\partial t} = -a\,\nabla p \qquad \mathrm{div}\mathbf{u} + b\frac{\partial p}{\partial t} = 0 \qquad \longrightarrow \qquad \frac{\partial^2 p}{\partial t^2} - \frac{a}{b}\,\Delta p = \mathbf{0}$$

continuity of p and of $\mathbf{u}\cdot\mathbf{n}$ at the interfaces

the actual medium is inhomogeneous

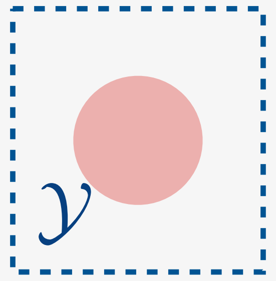
We end up with an effective wave equation

$$\frac{\partial \mathbf{U}}{\partial t} = -\mathsf{a}_{\mathrm{eff}}\,\nabla P \qquad \mathrm{div}\mathbf{U} + b_{\mathrm{eff}}\frac{\partial P}{\partial t} = 0 \qquad \longrightarrow \qquad \frac{\partial^2 P}{\partial t^2} - \mathrm{div}\left(\frac{1}{b_{\mathrm{eff}}}\mathsf{a}_{\mathrm{eff}}\,\nabla P\right)$$

same structure of the equations
the effective medium is homogeneous
but anisotropic in general

The price to pay: solve « cell problems » on $(Q_i, \mathbf{V}_i)$ *i= 1, 2 (2d), i=1,2,3 (3d),* solutions to:

$\mathrm{div}\mathbf{V_i} = 0, \quad \mathbf{V_i} = a(\mathbf{e}_i + Q_i)$
$Q_i, \mathbf{V_i}$ periodic
$Q_i, \mathbf{V_i}\cdot\mathbf{n}$ continuous



which provides

$$\mathsf{a}_{\mathrm{eff}} = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}$$

$$a_{ij} = \int_{\mathcal{Y}} a\left(\delta_{ij} + \frac{\partial Q_i}{\partial y_j}\right)$$

and we simply have

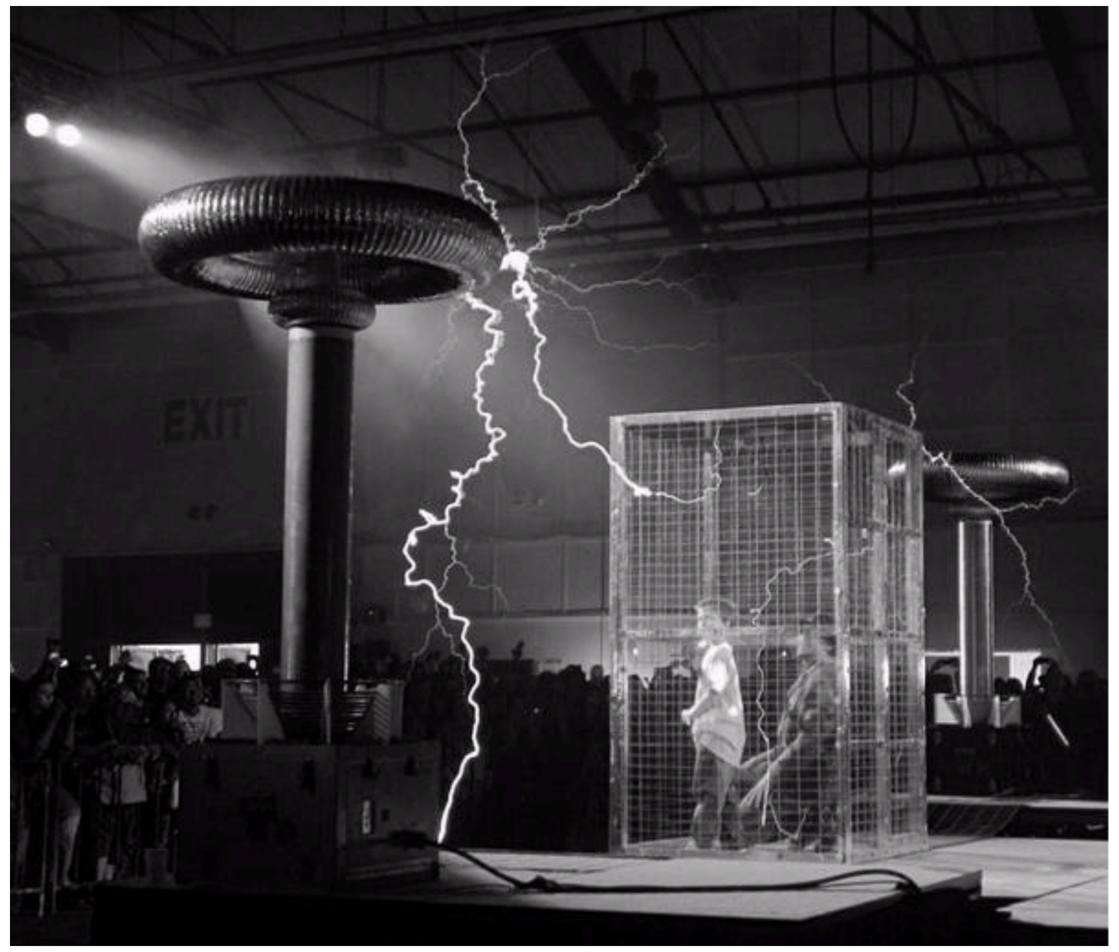
$$b_{\mathrm{eff}} = \int_{\mathcal{Y}} b$$

the anisotropy is given by aeff

Homogenization

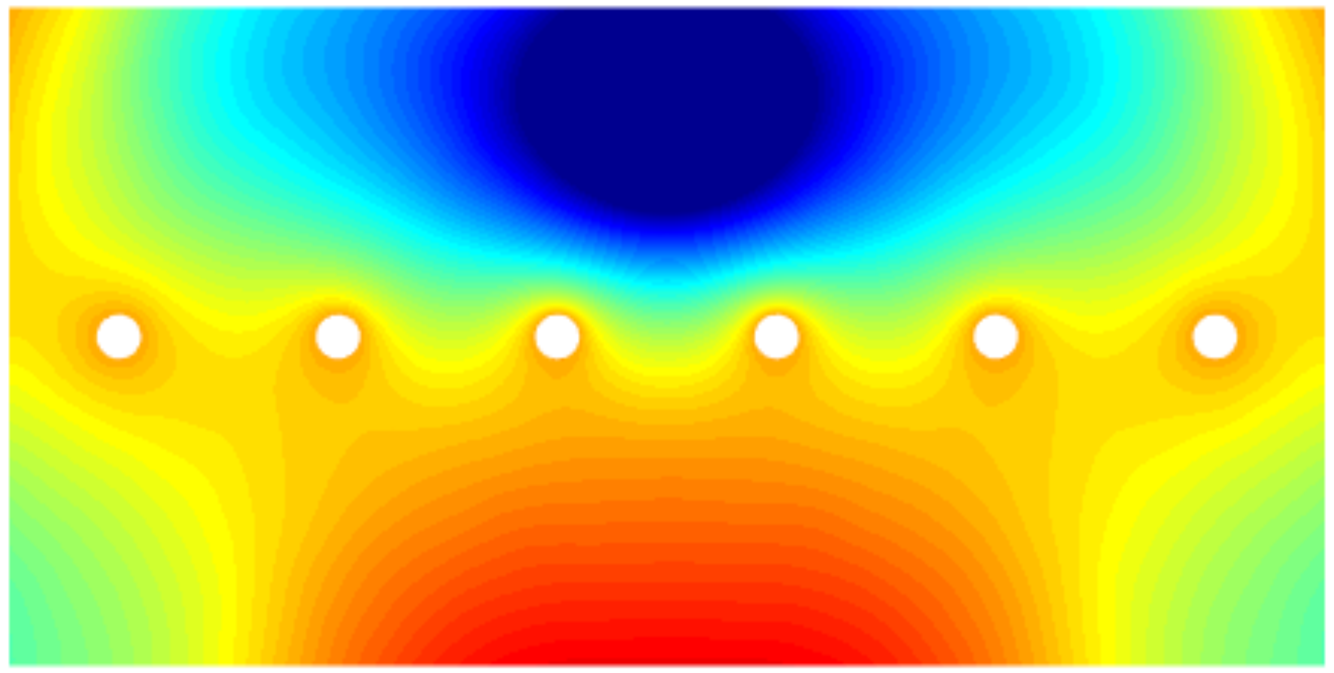
Examples of applications

that require more technicality

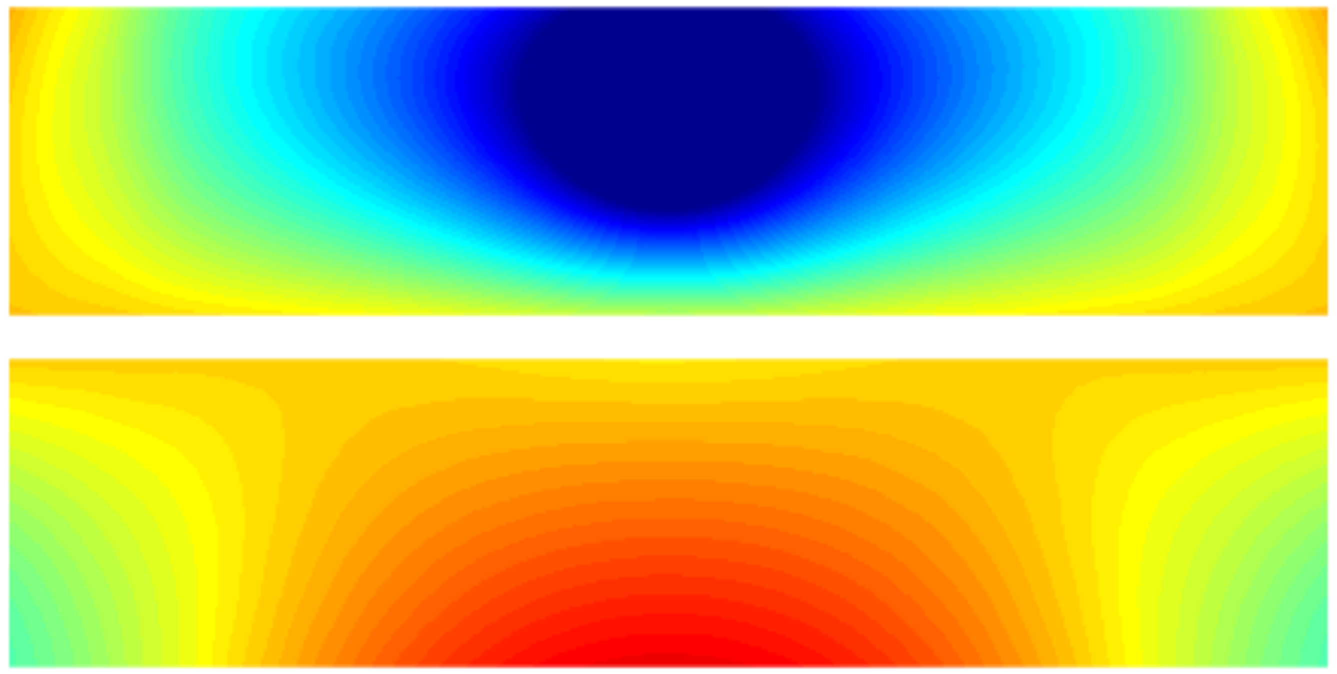


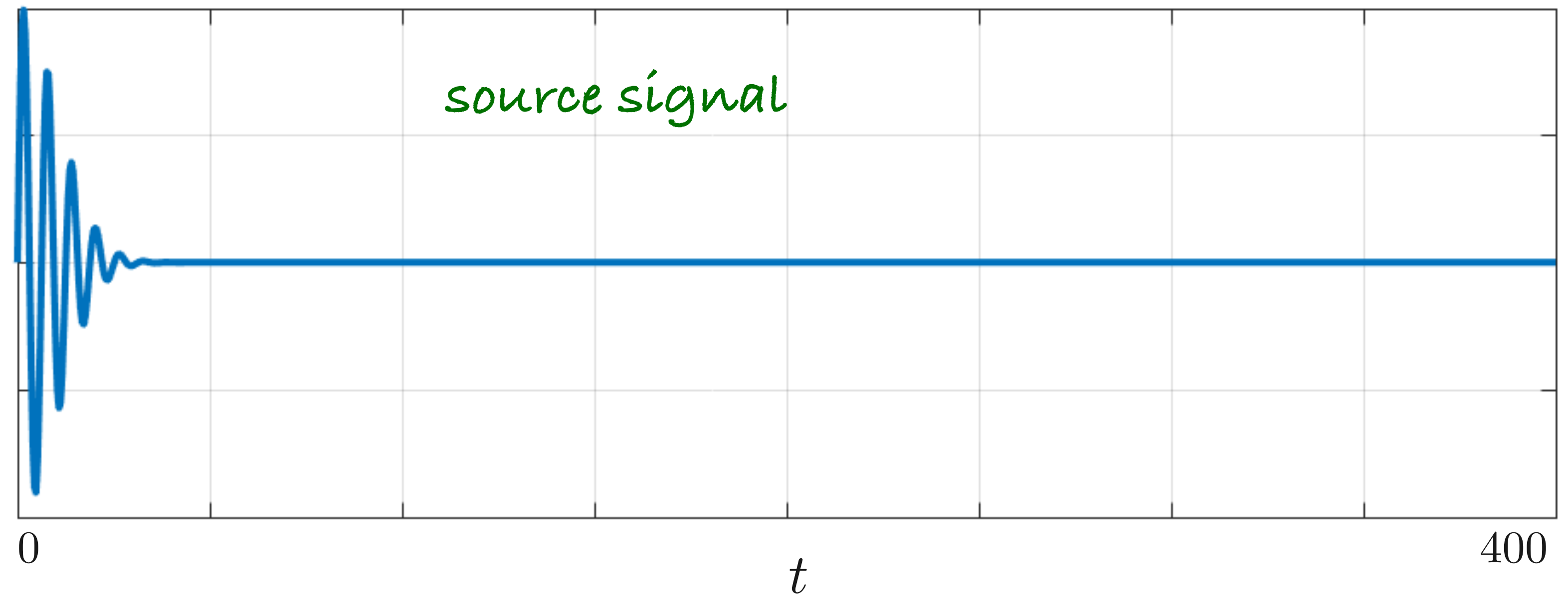
Faraday cage

real problem

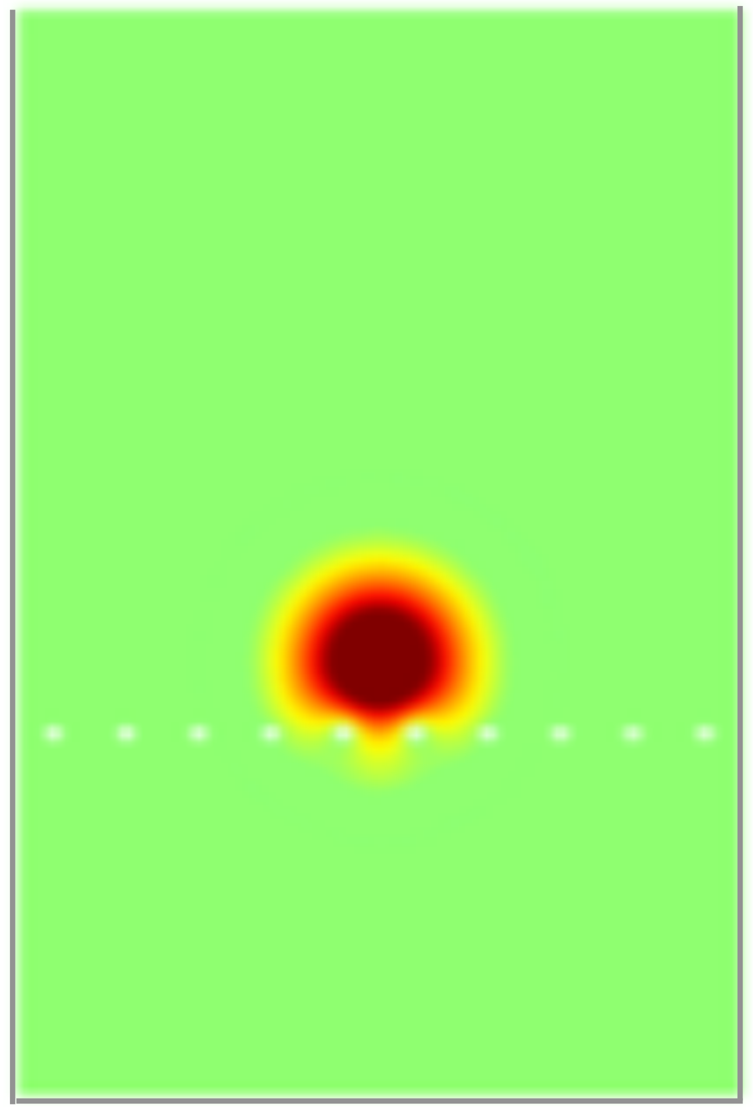


homogenized problem

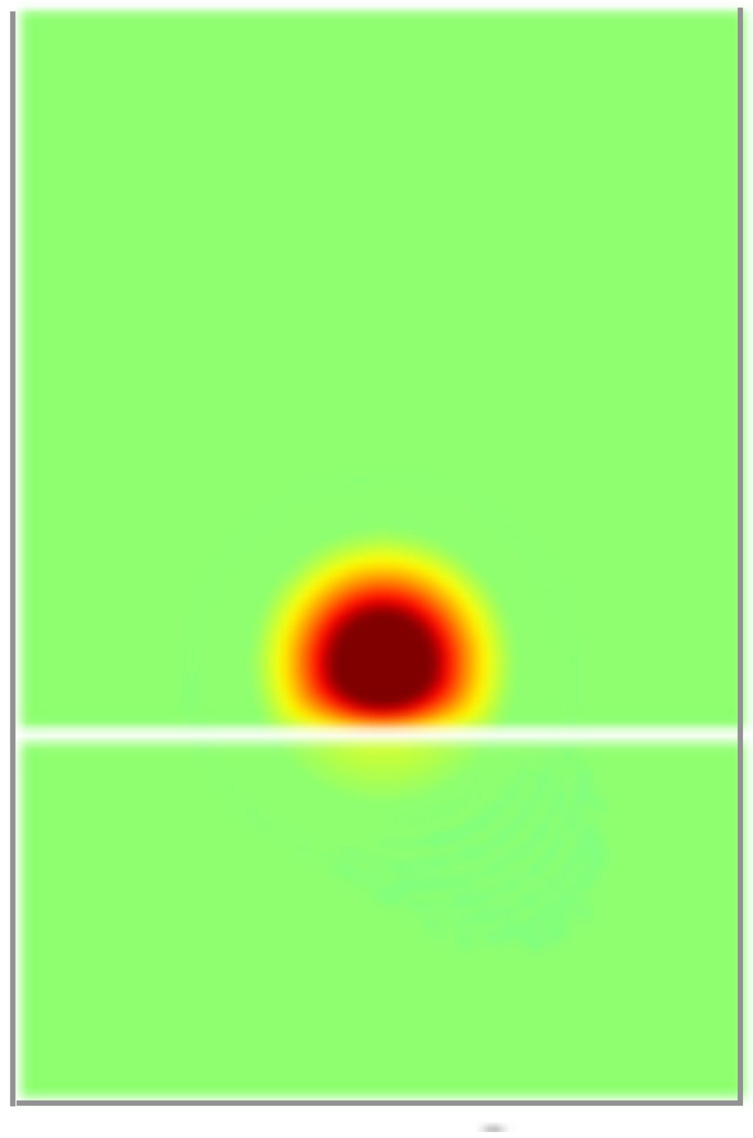




real problem



homogenized problem



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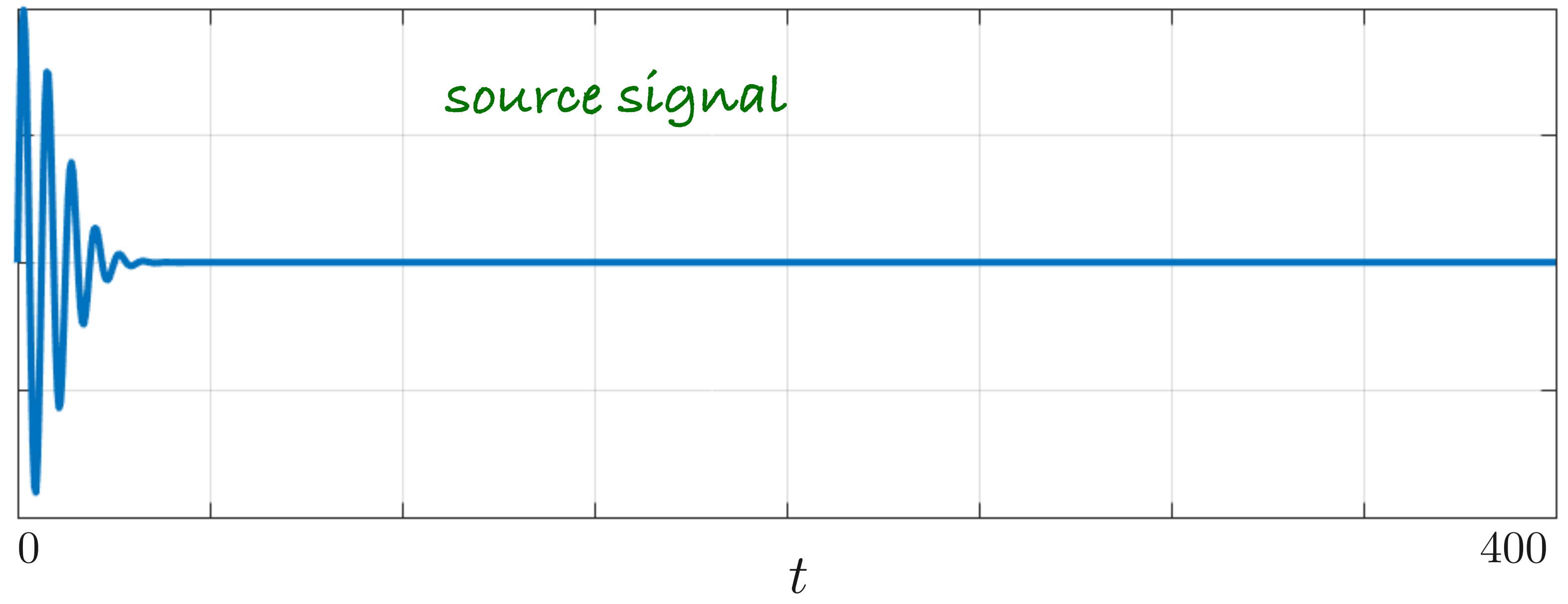
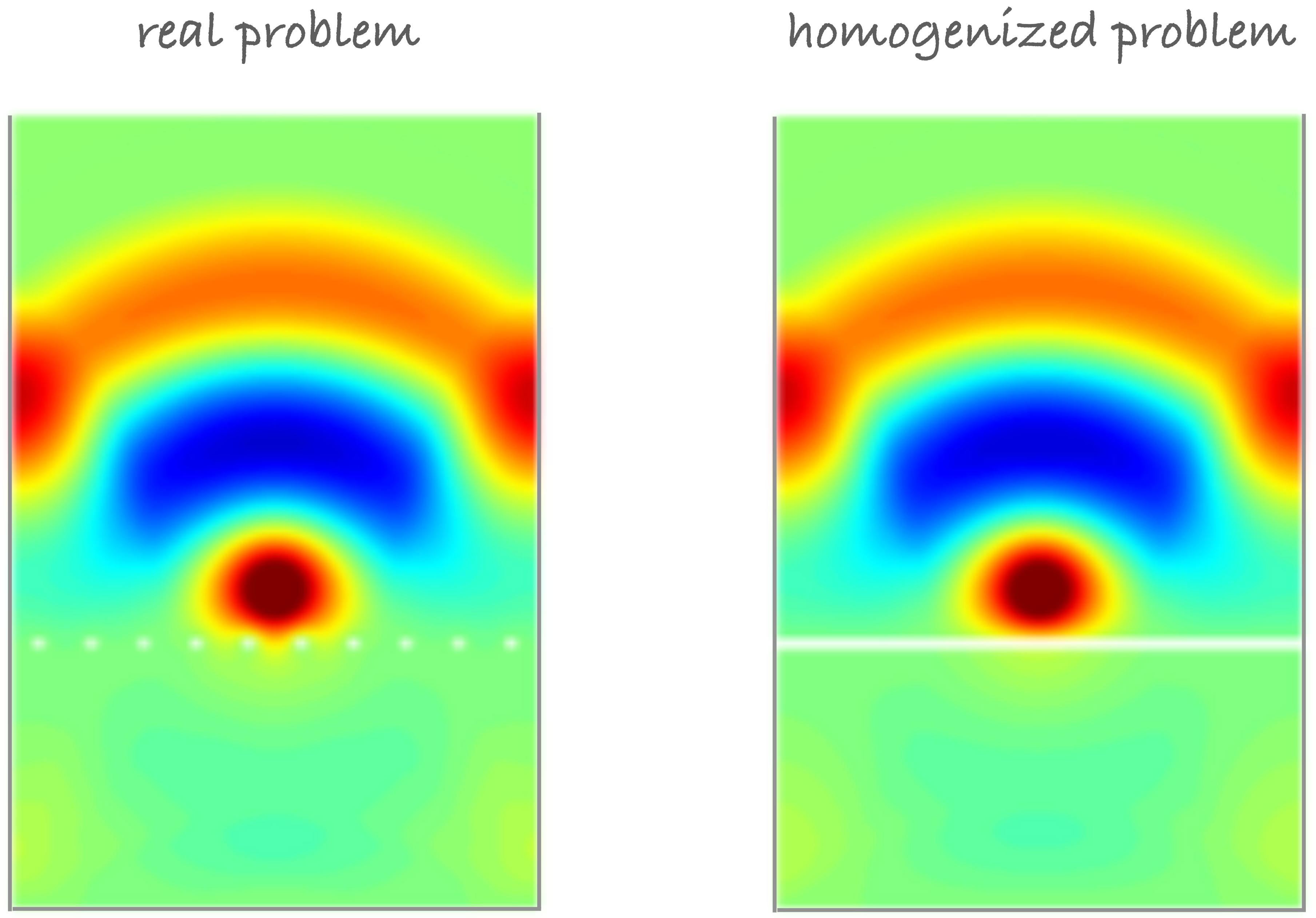
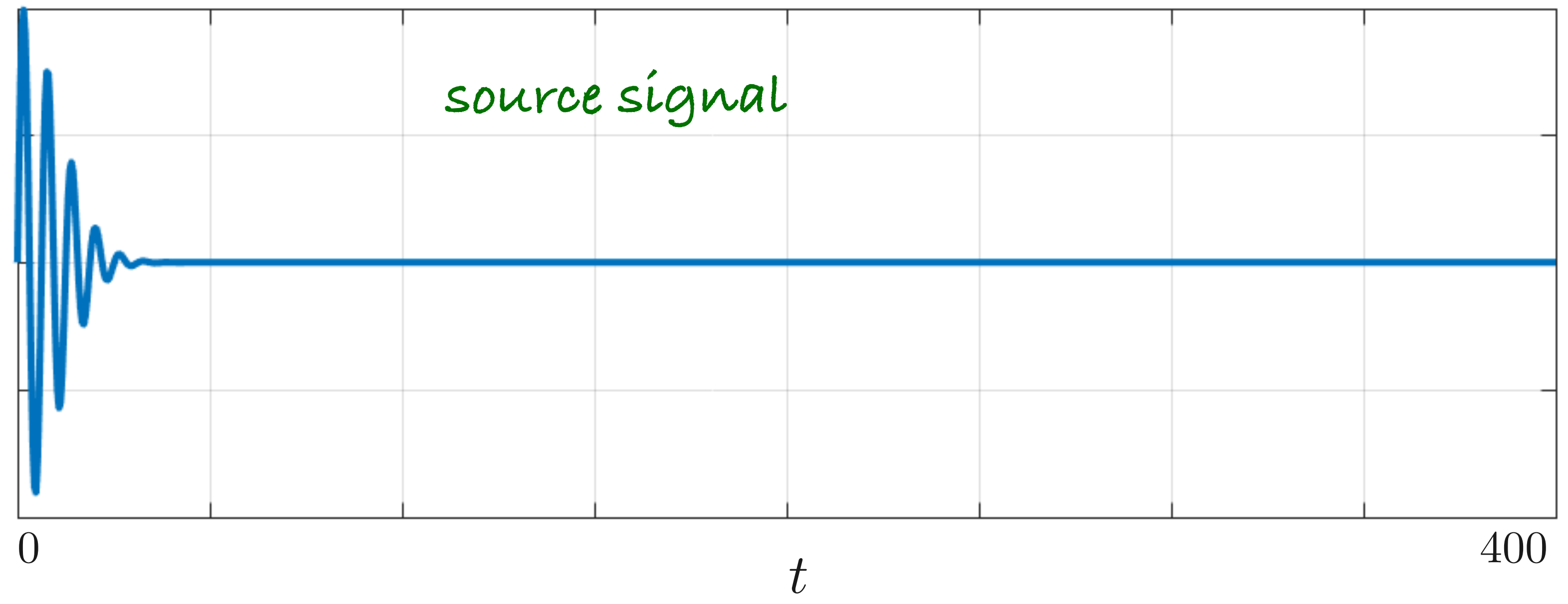
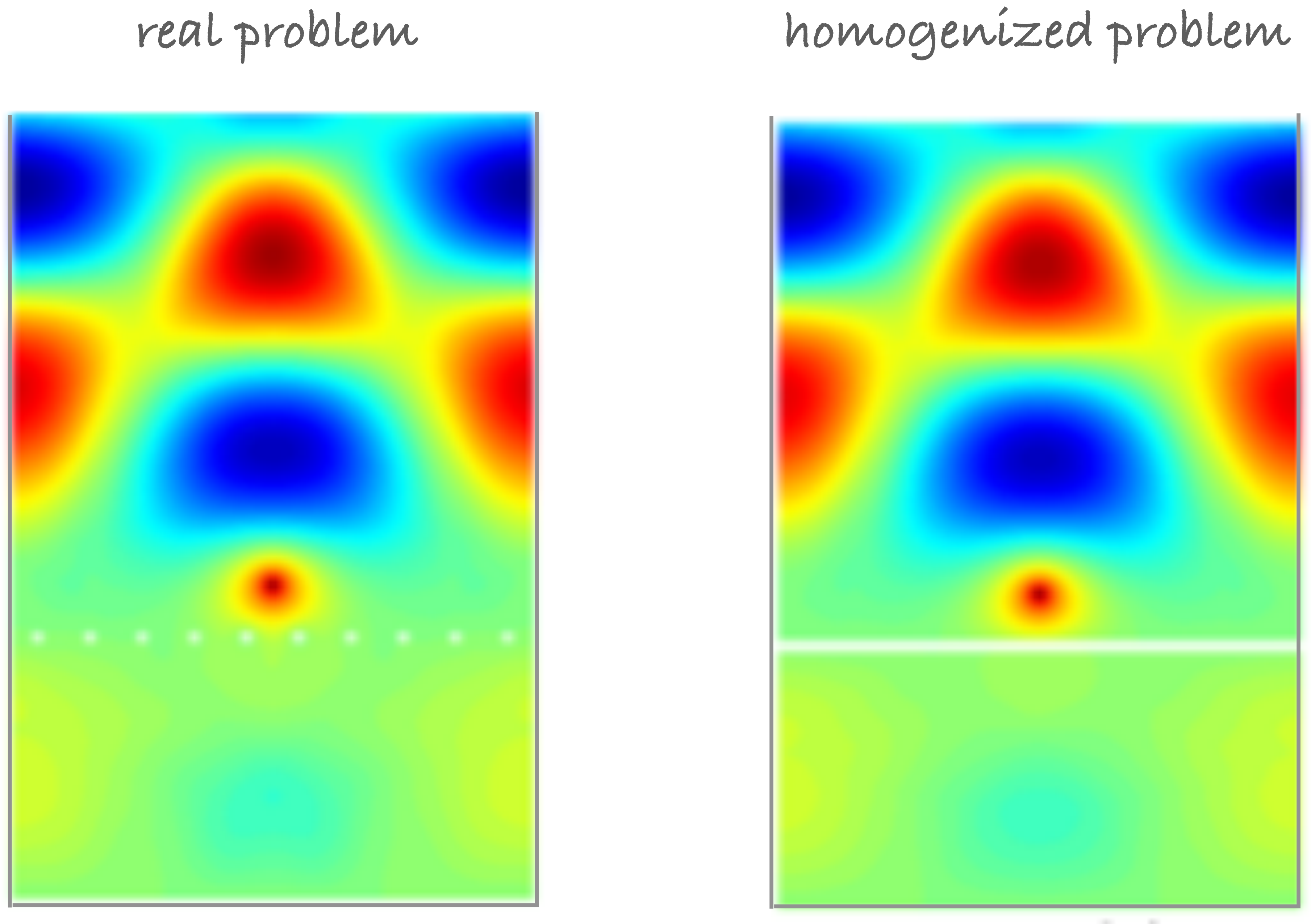


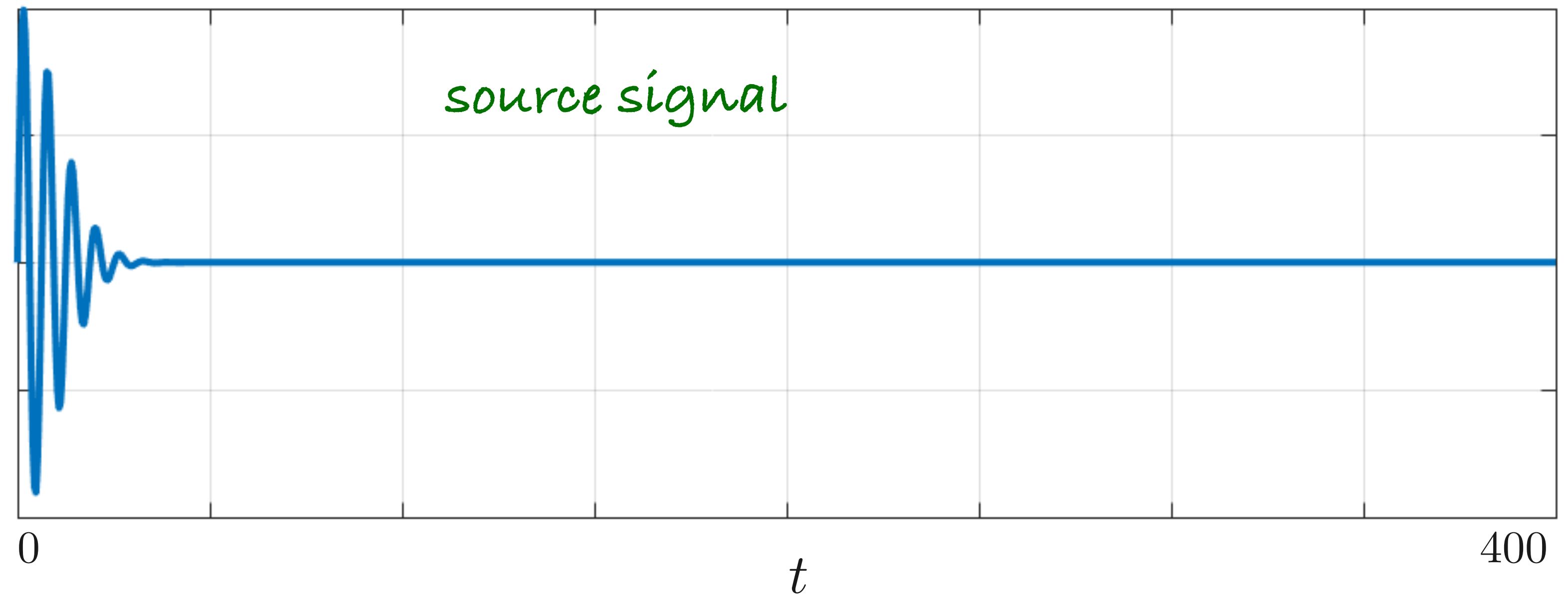


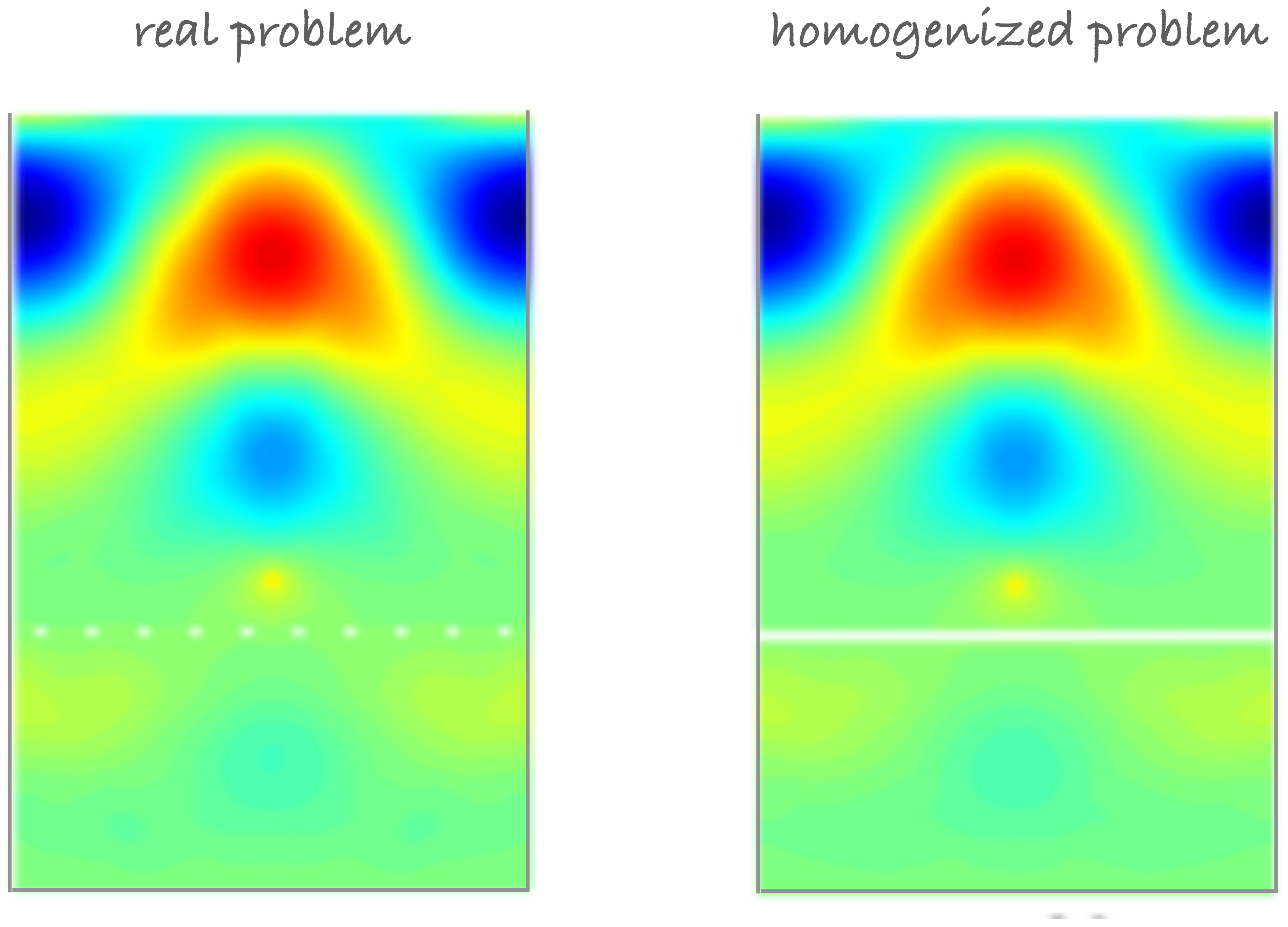
Faraday cage



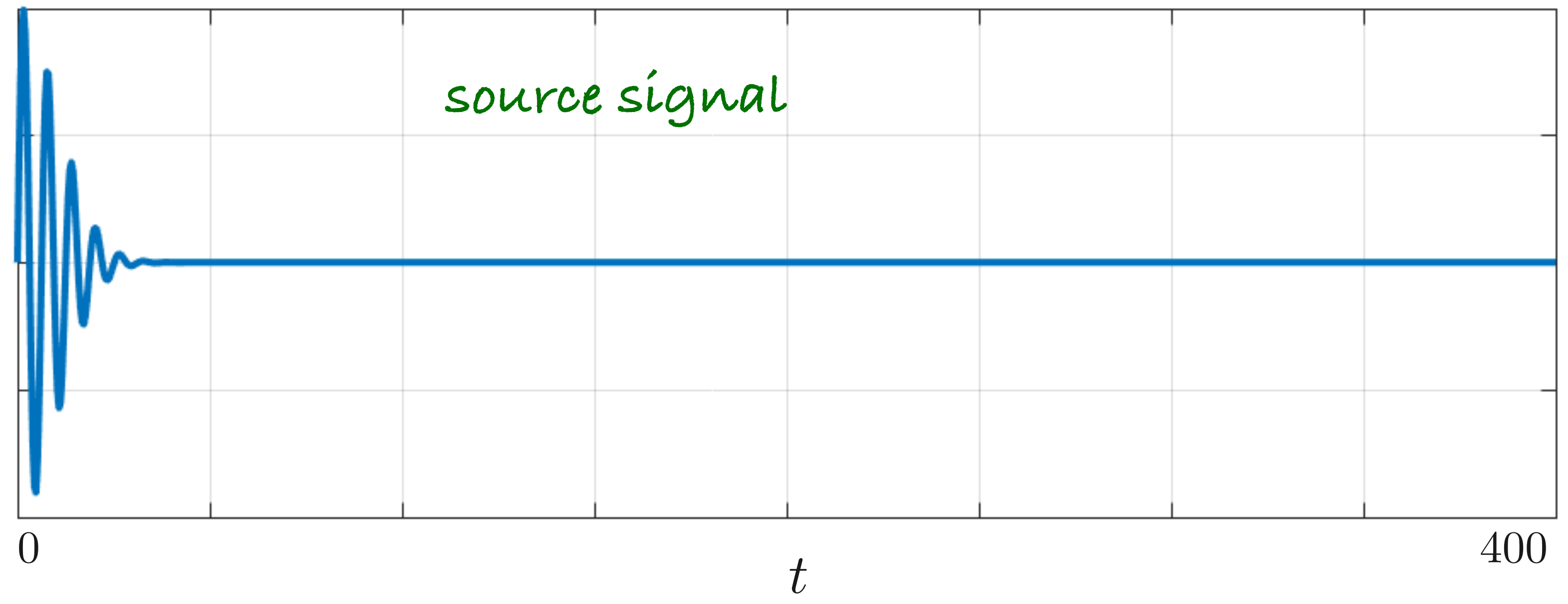


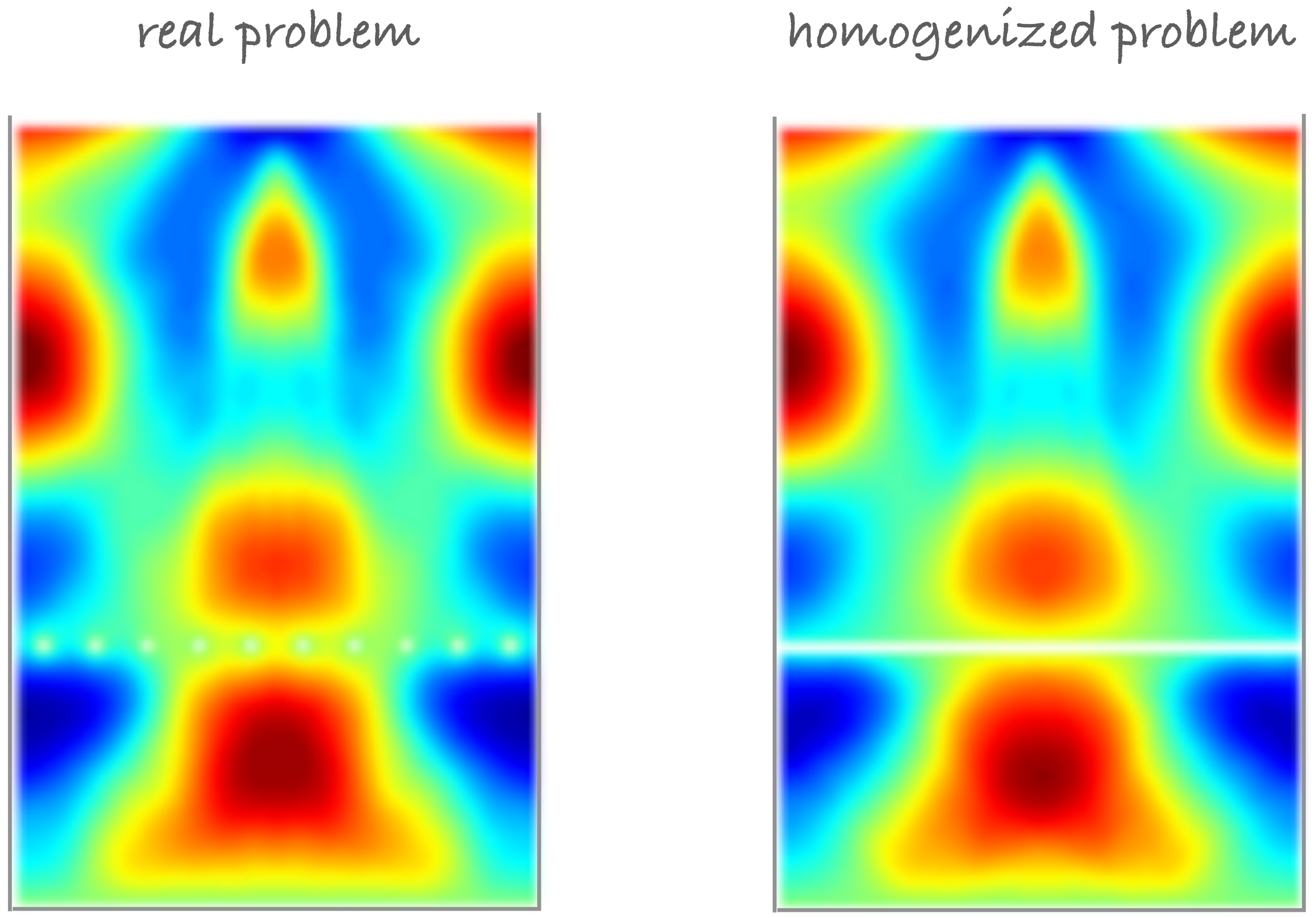
Faraday cage





Faraday cage





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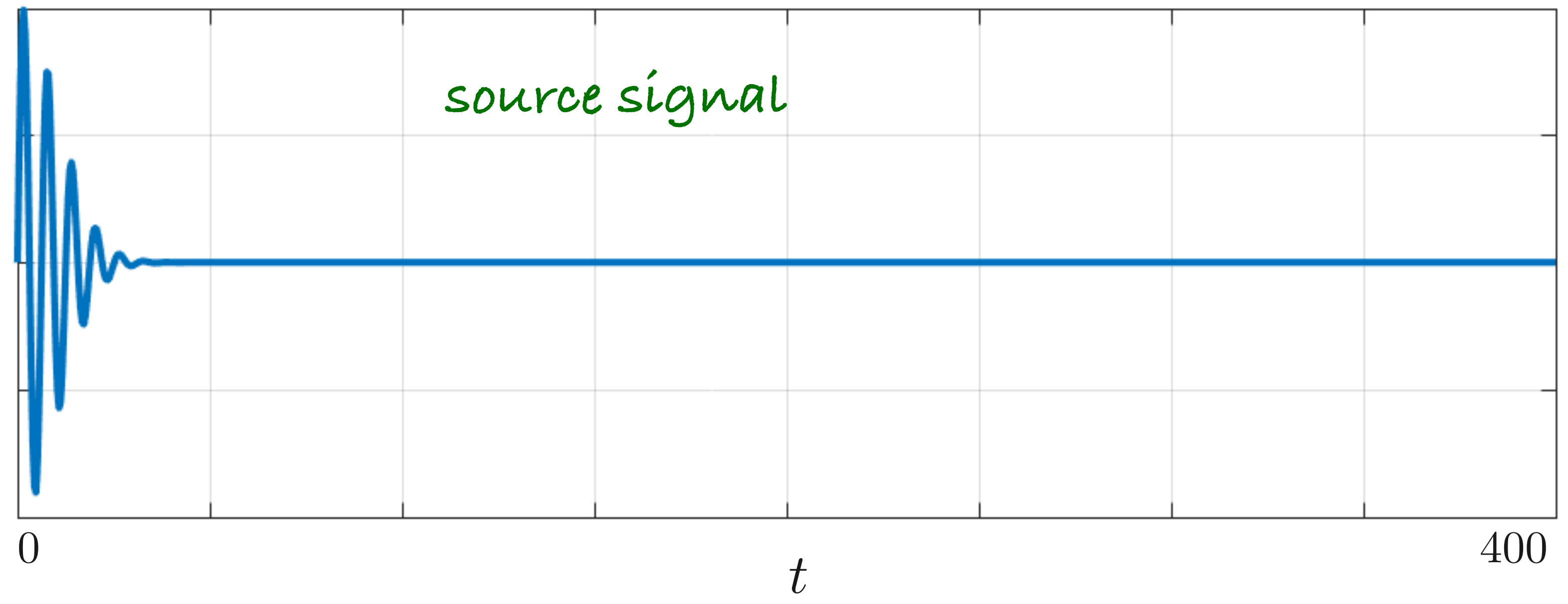
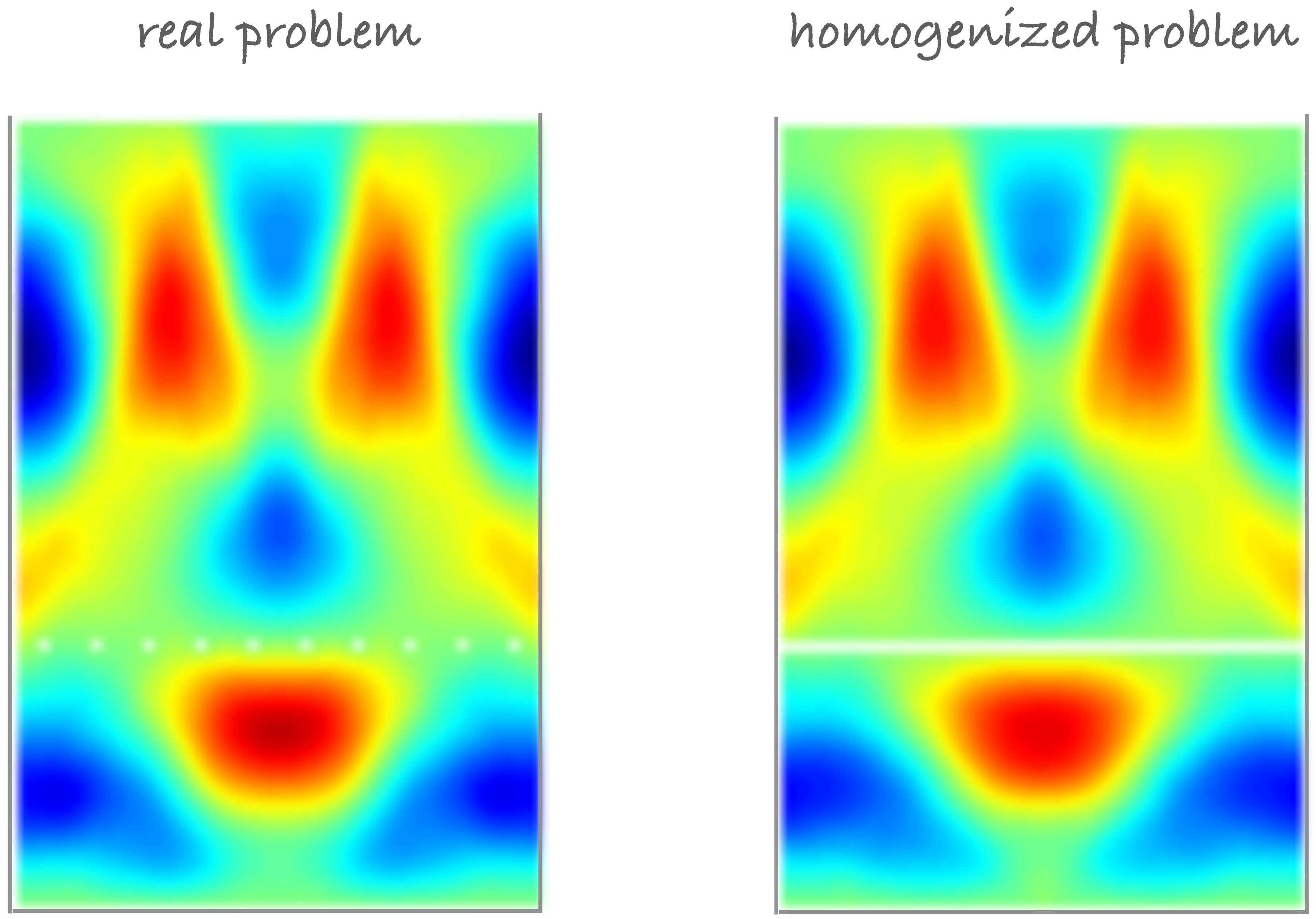
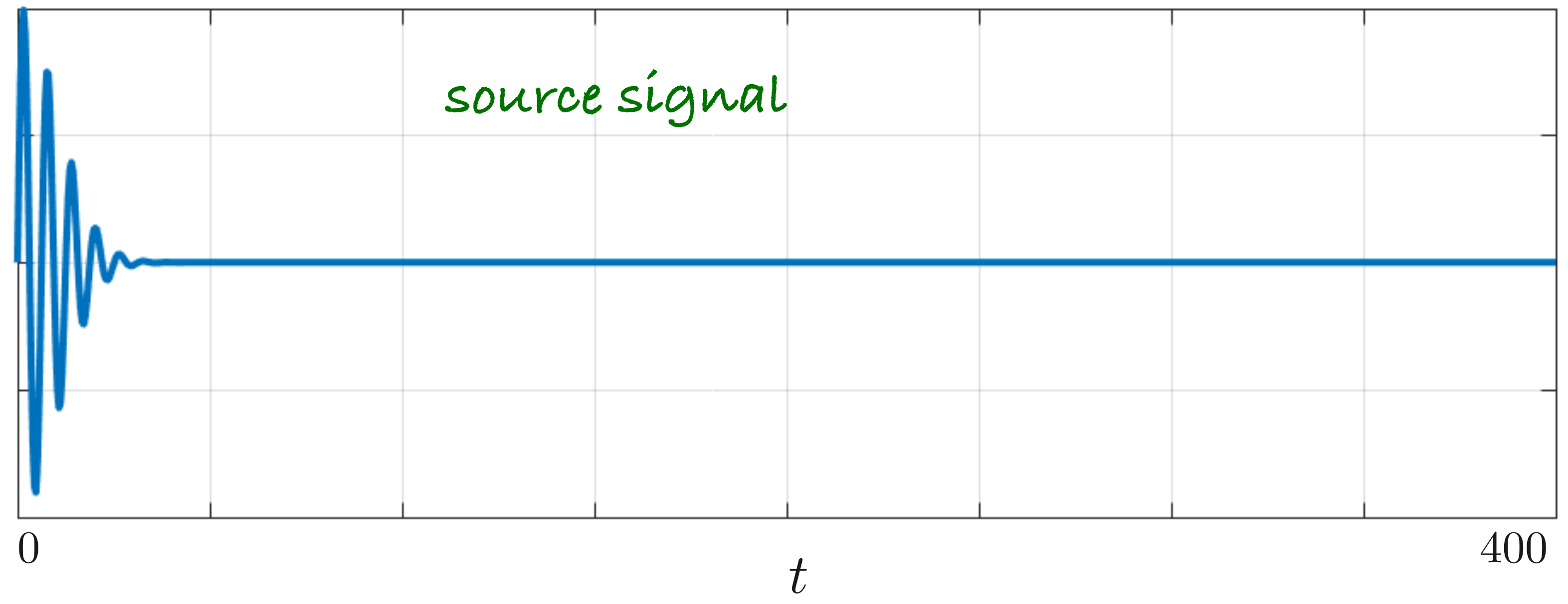
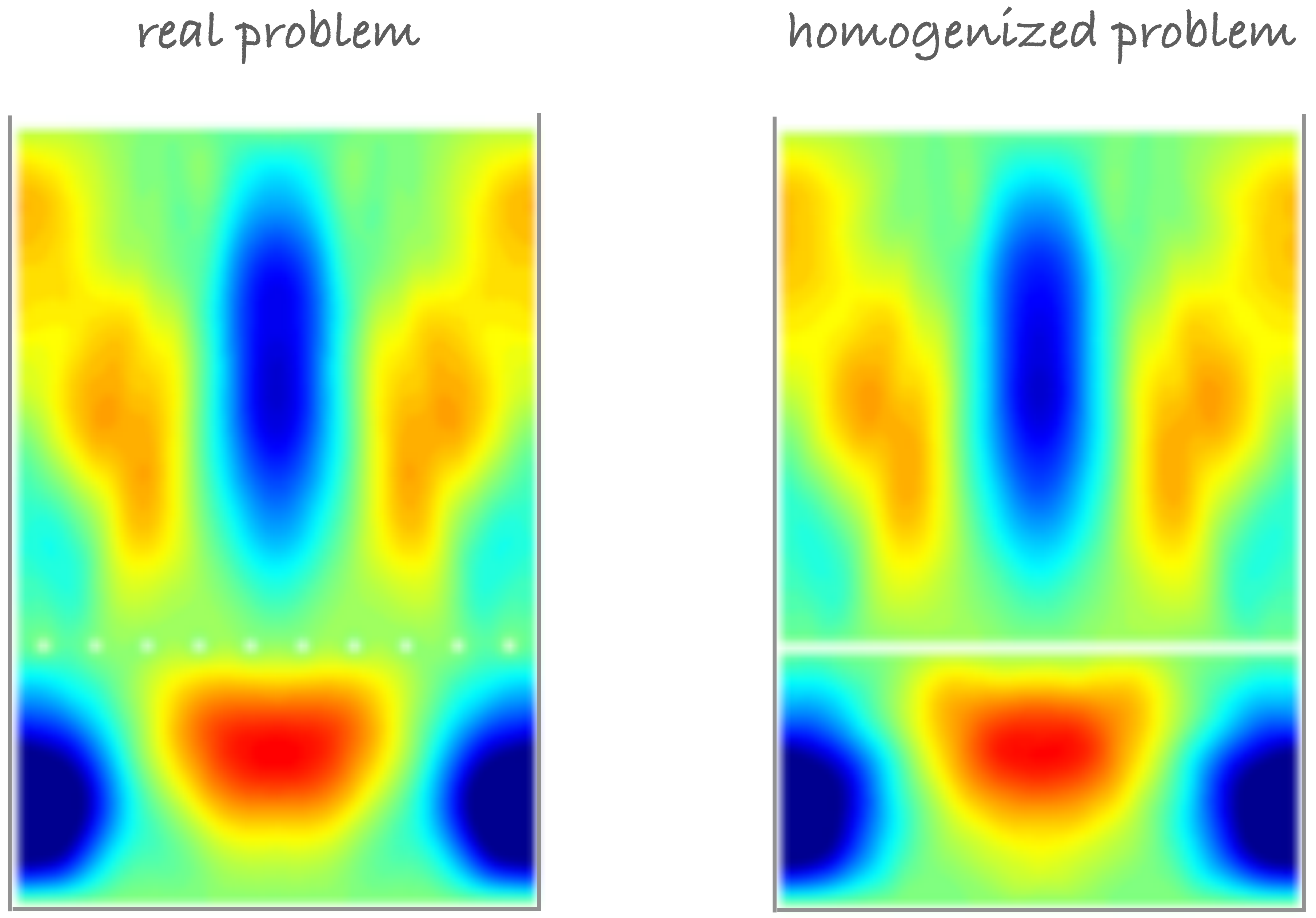


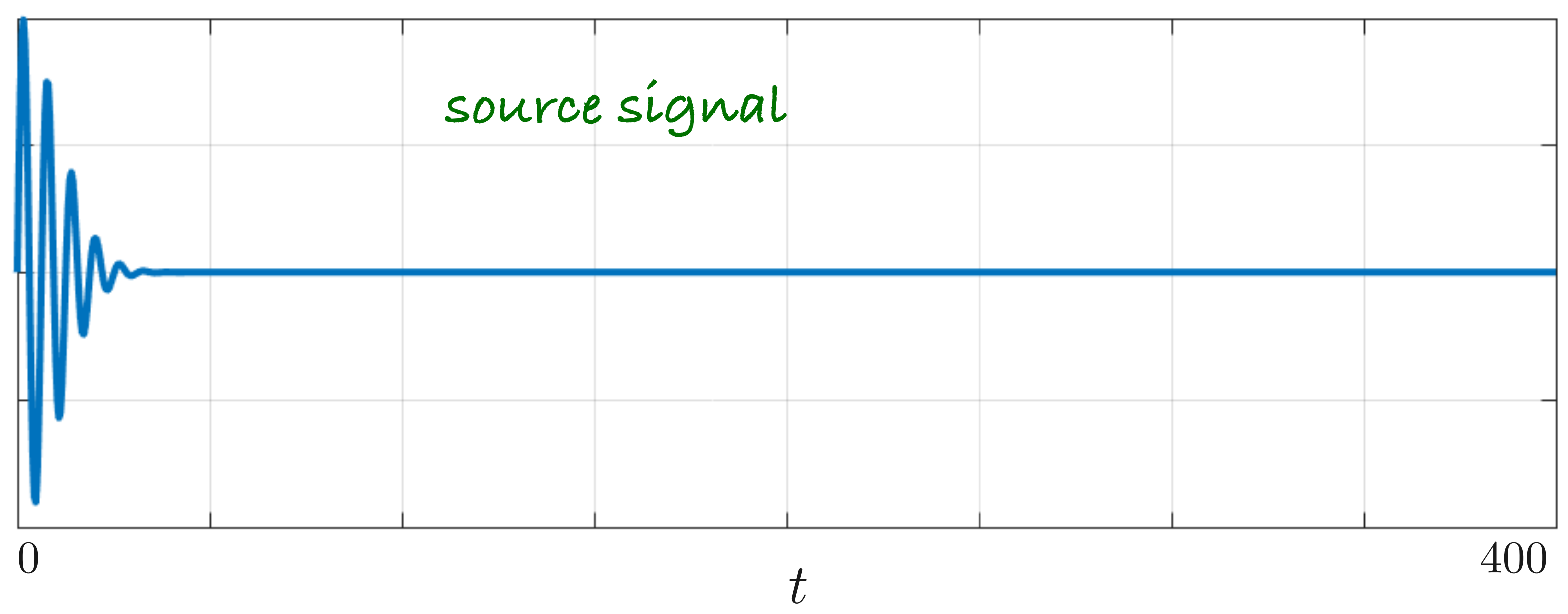


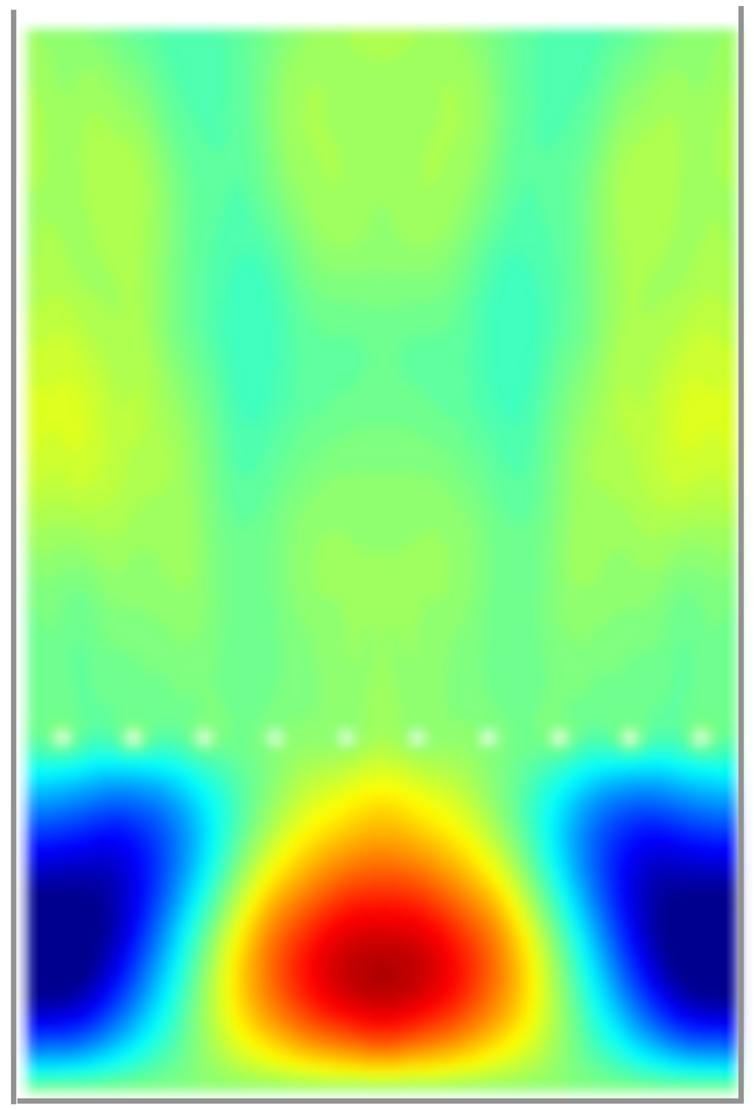
Faraday cage

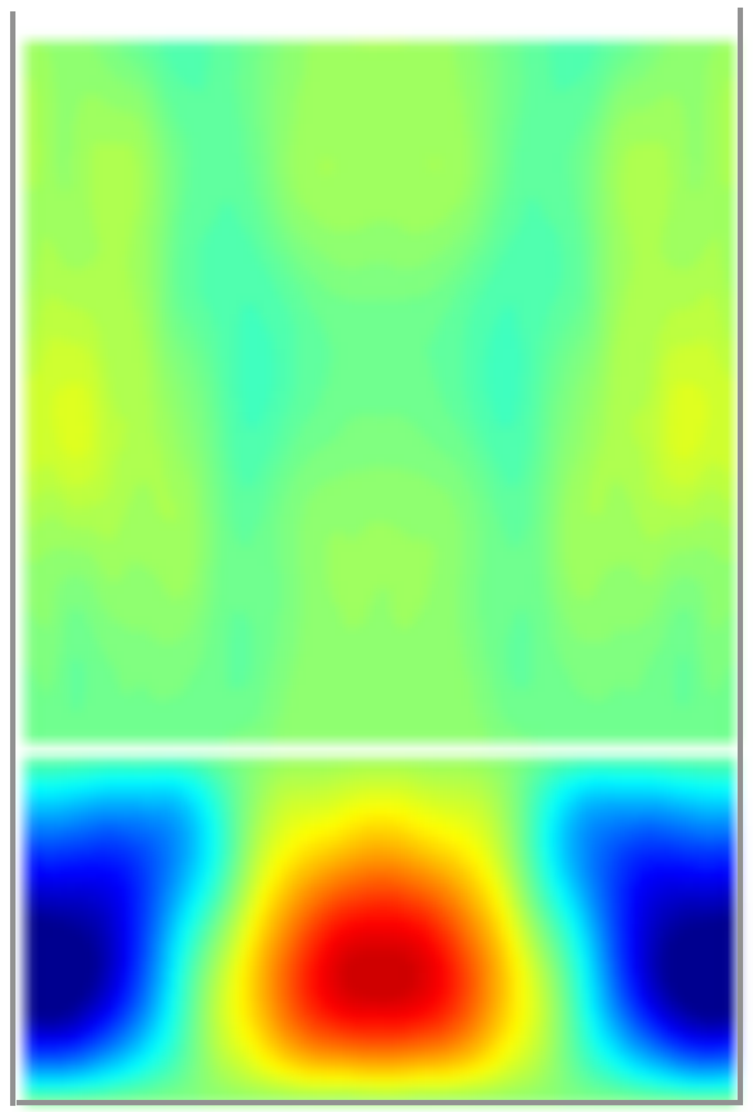




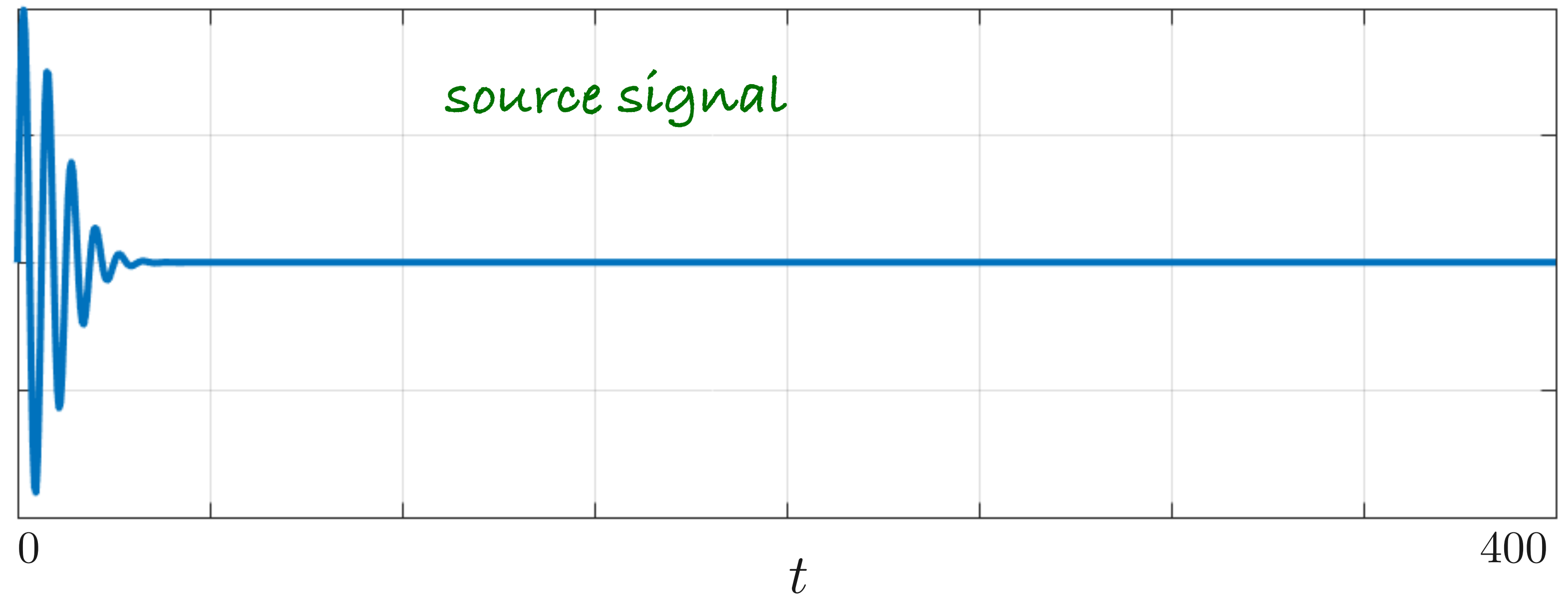
Faraday cage

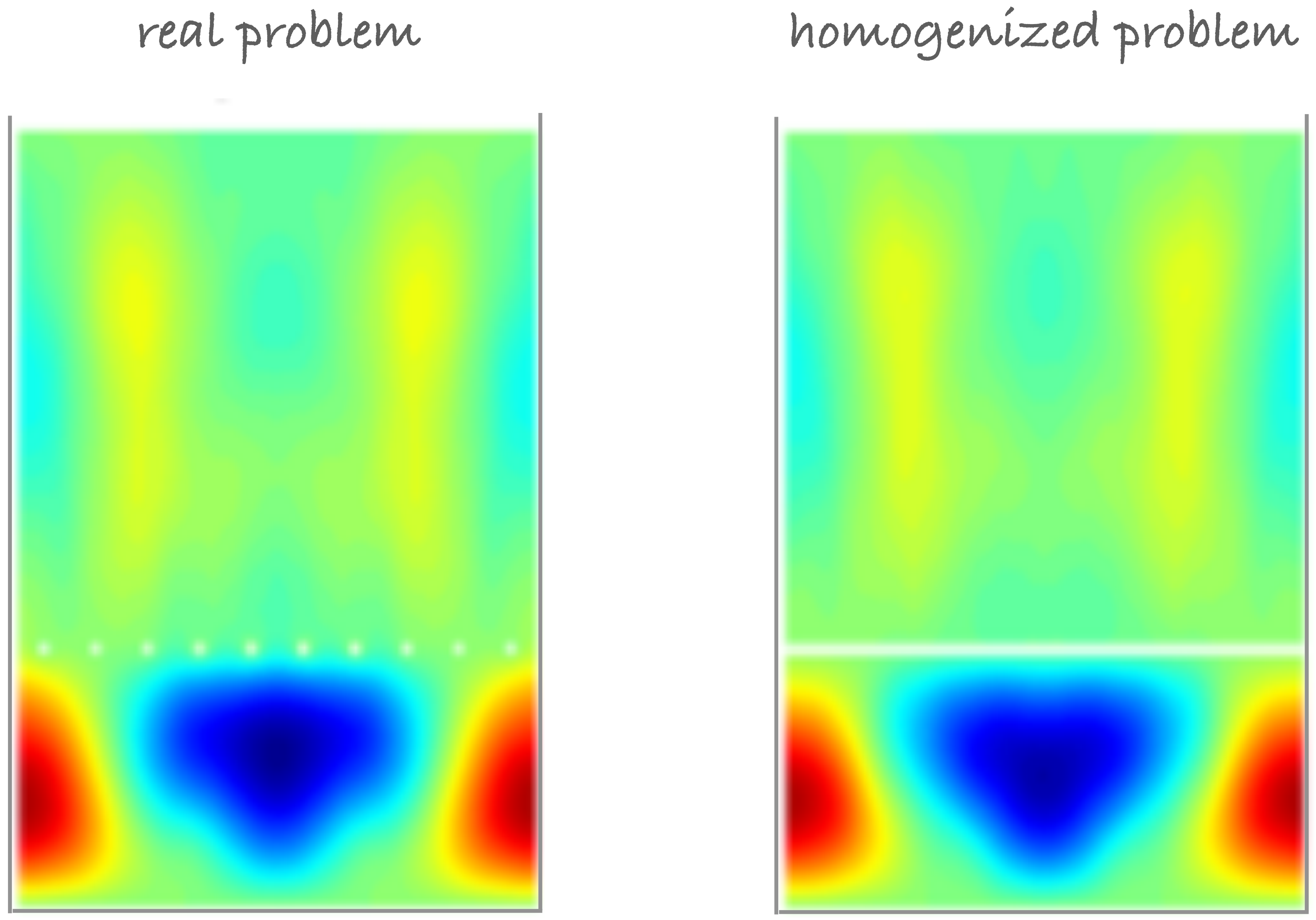




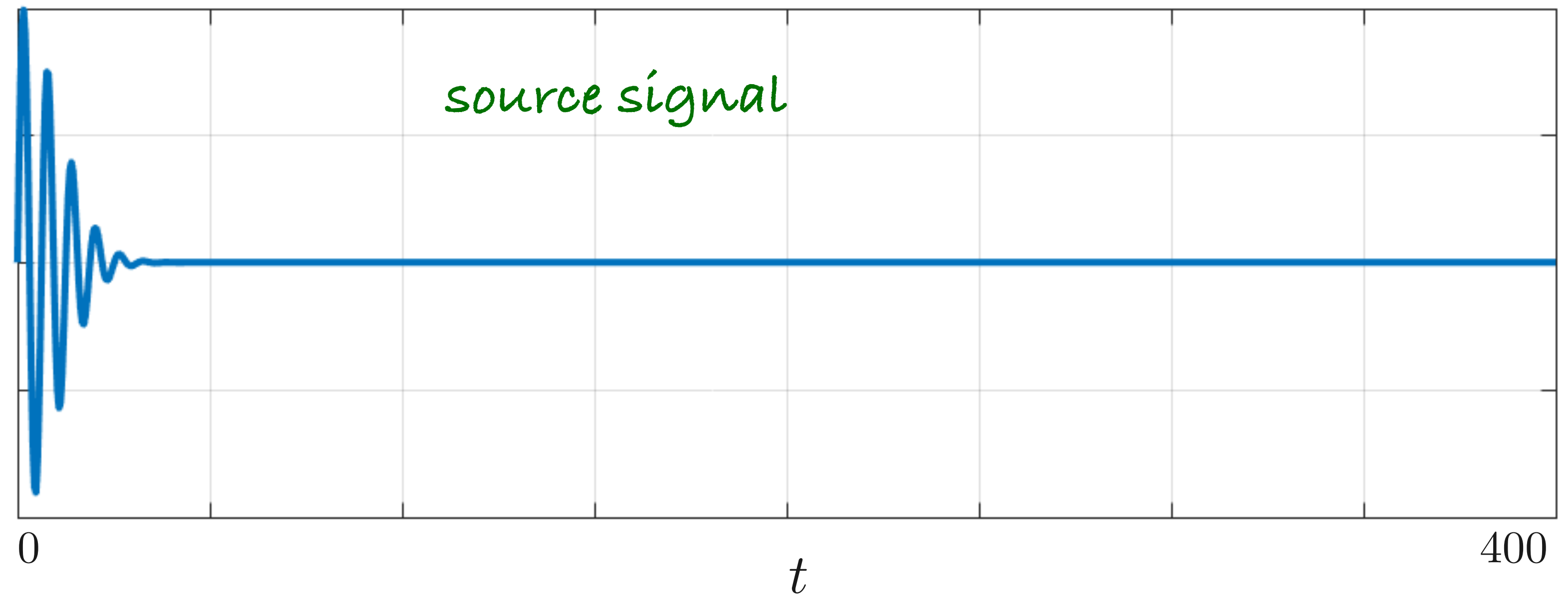


Faraday cage

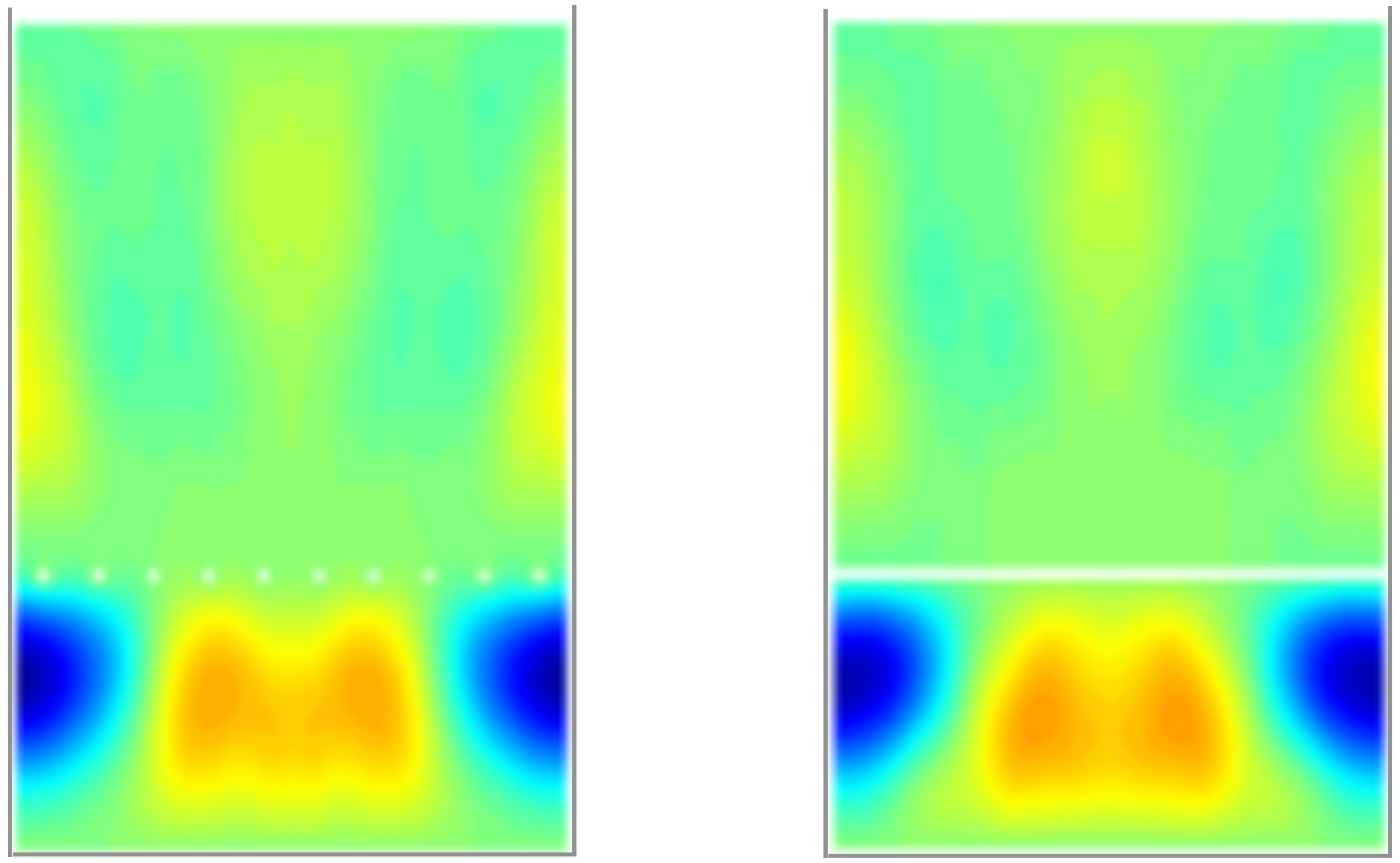




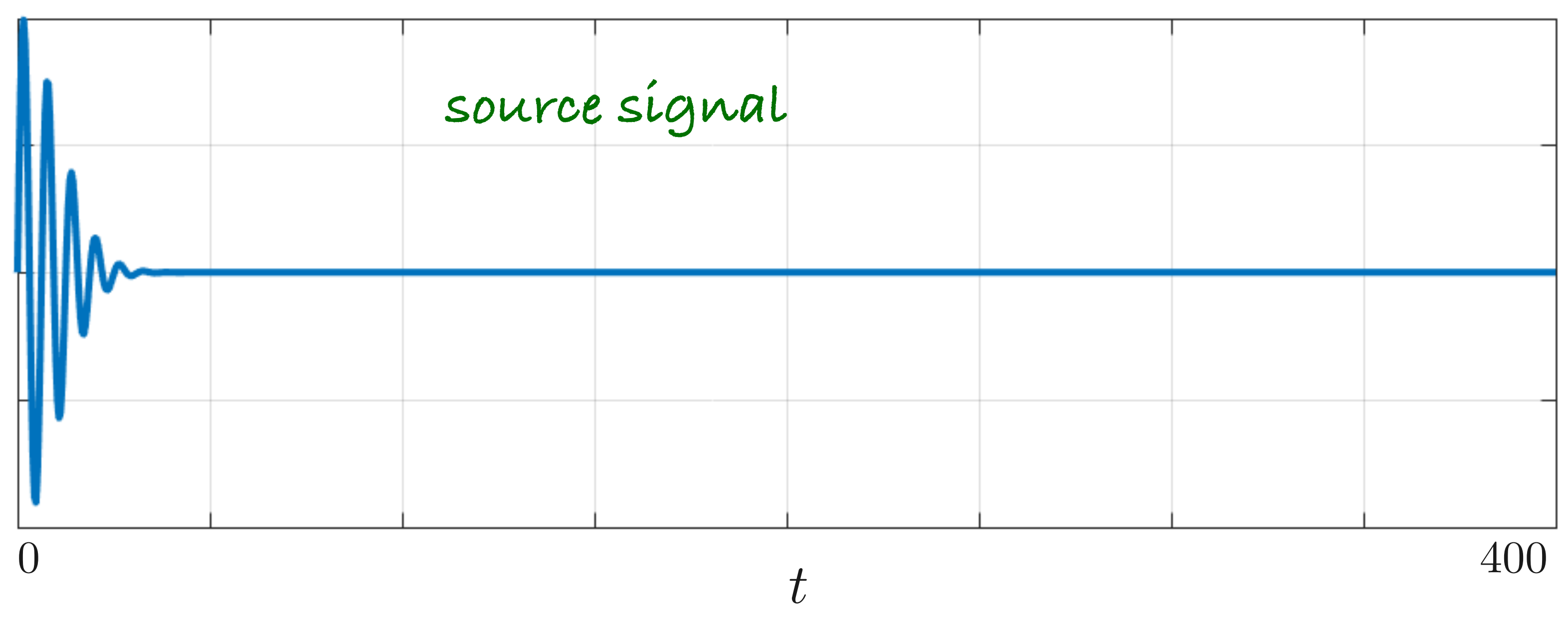
Faraday cage

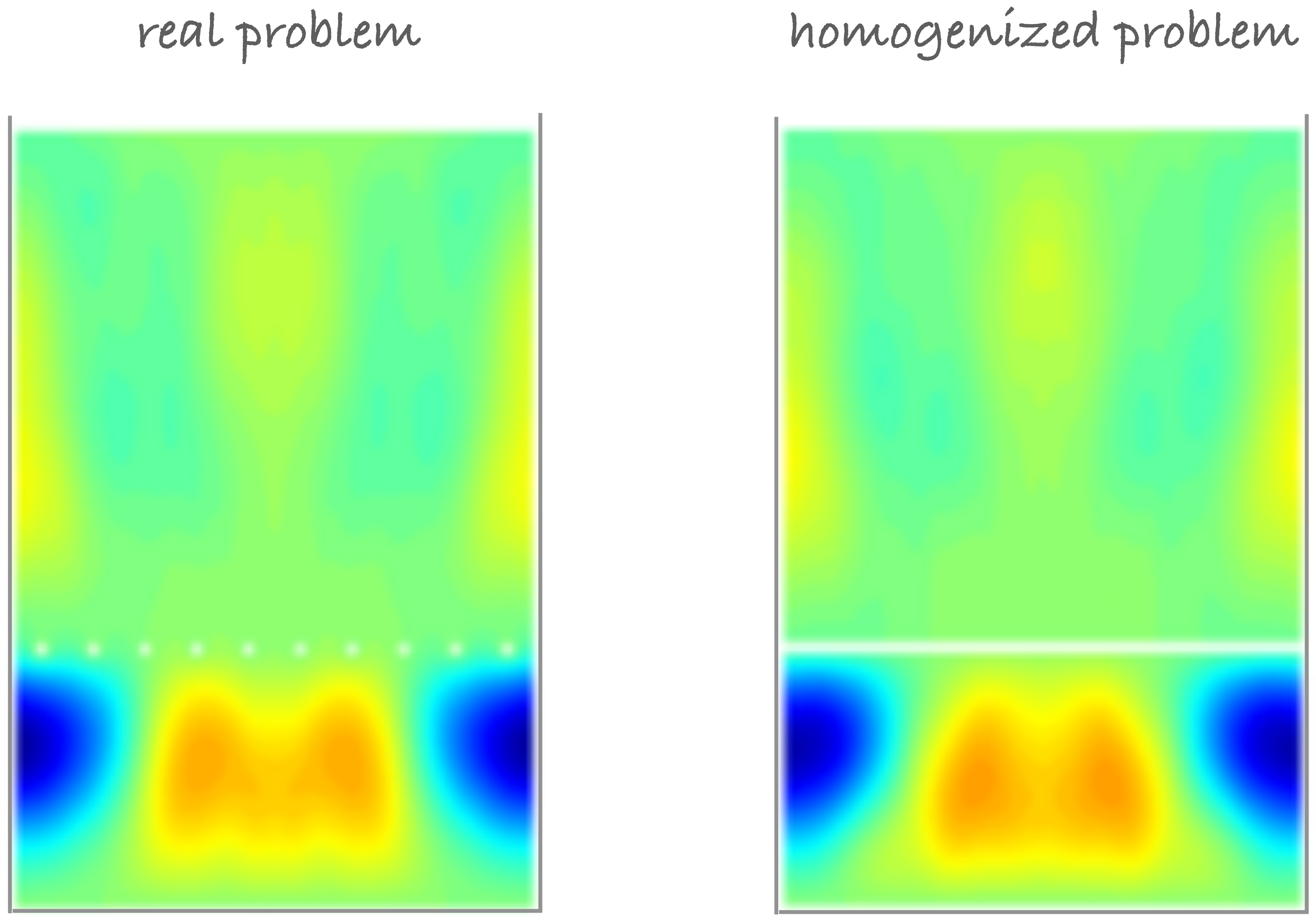


real problem   homogenized problem

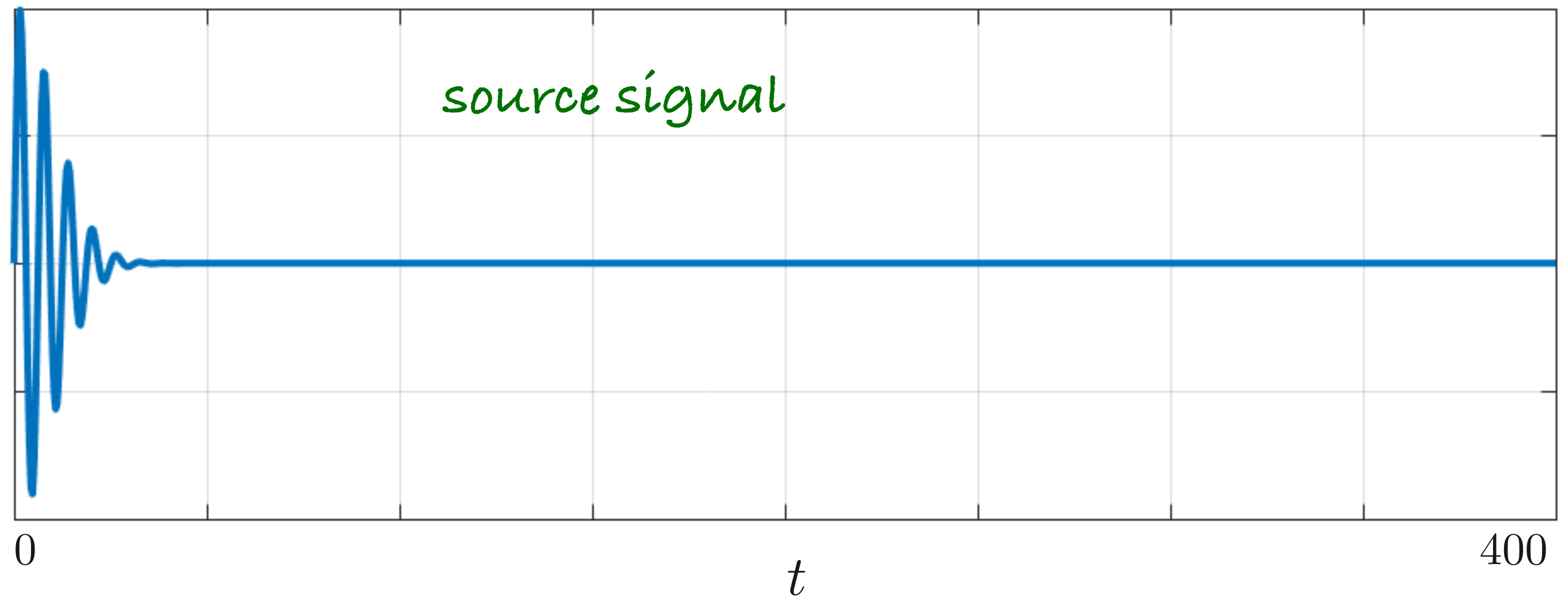


Faraday cage

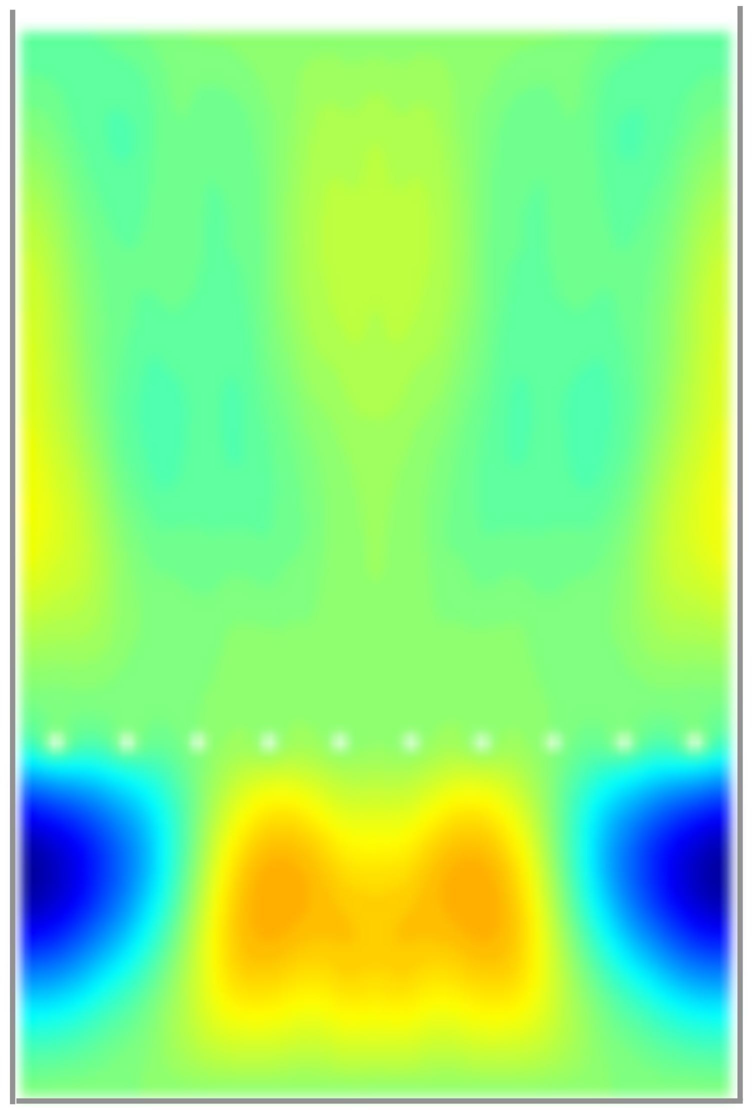




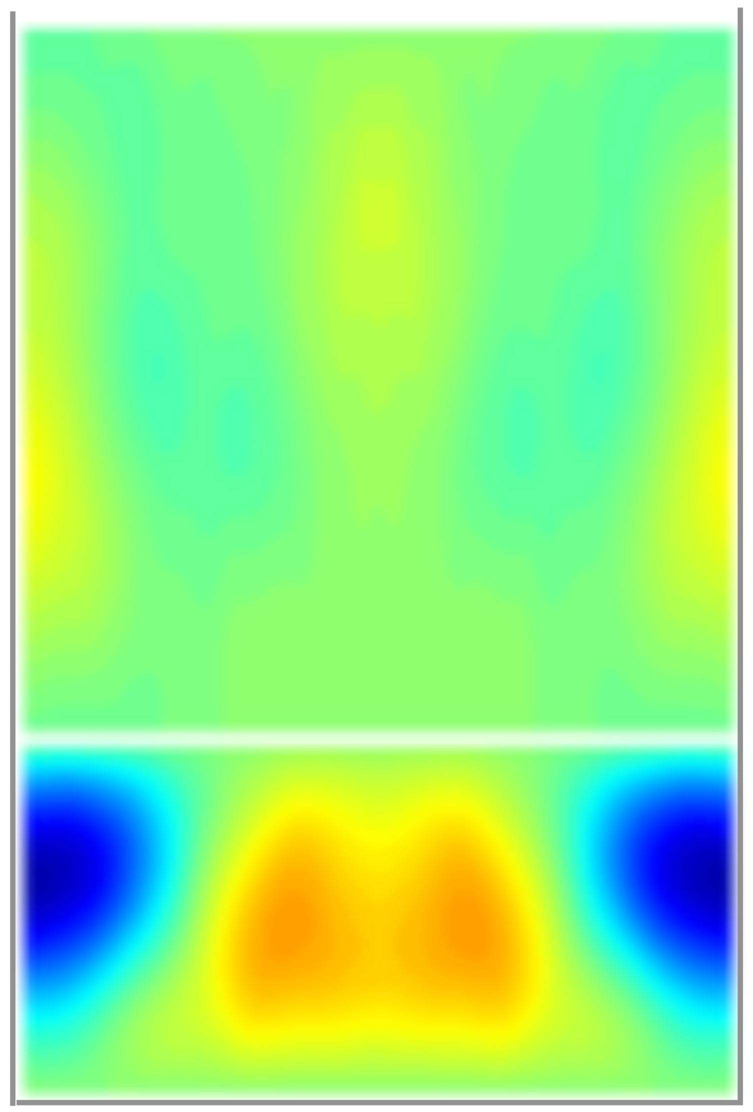
Faraday cage



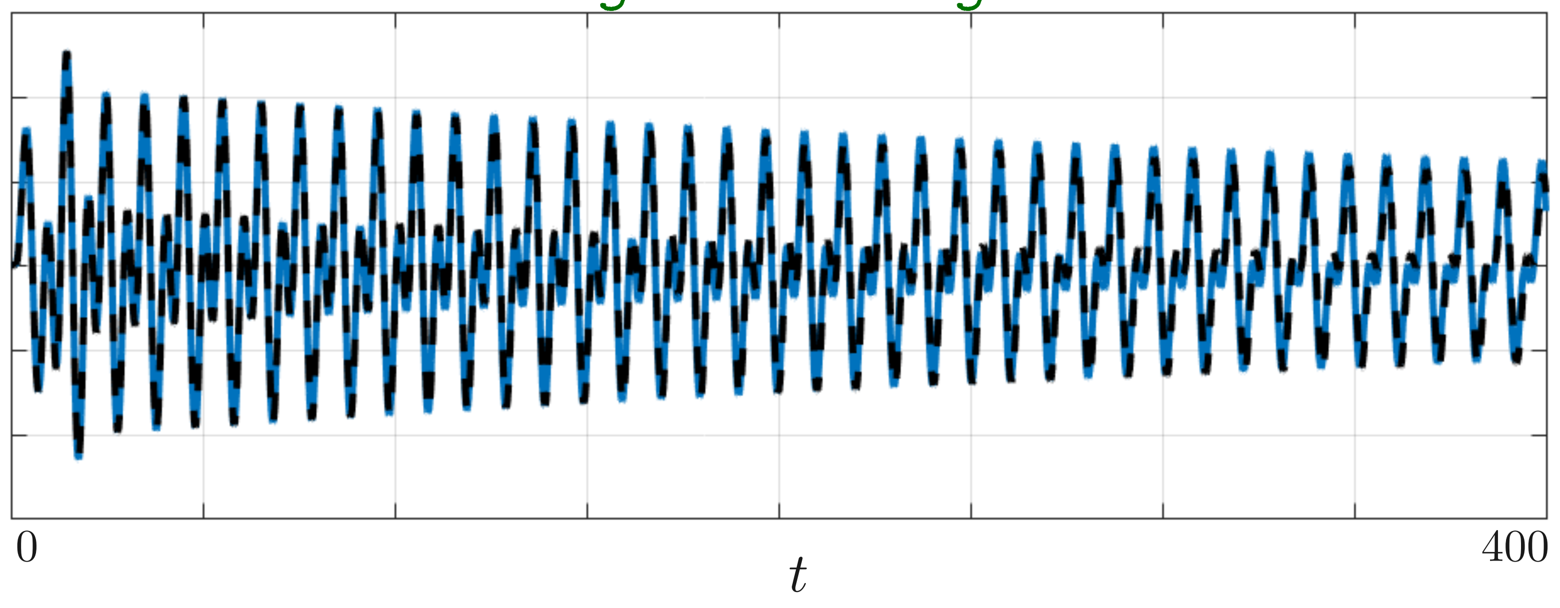
real problem



homogenized problem



signal in the cage



homogenized solution

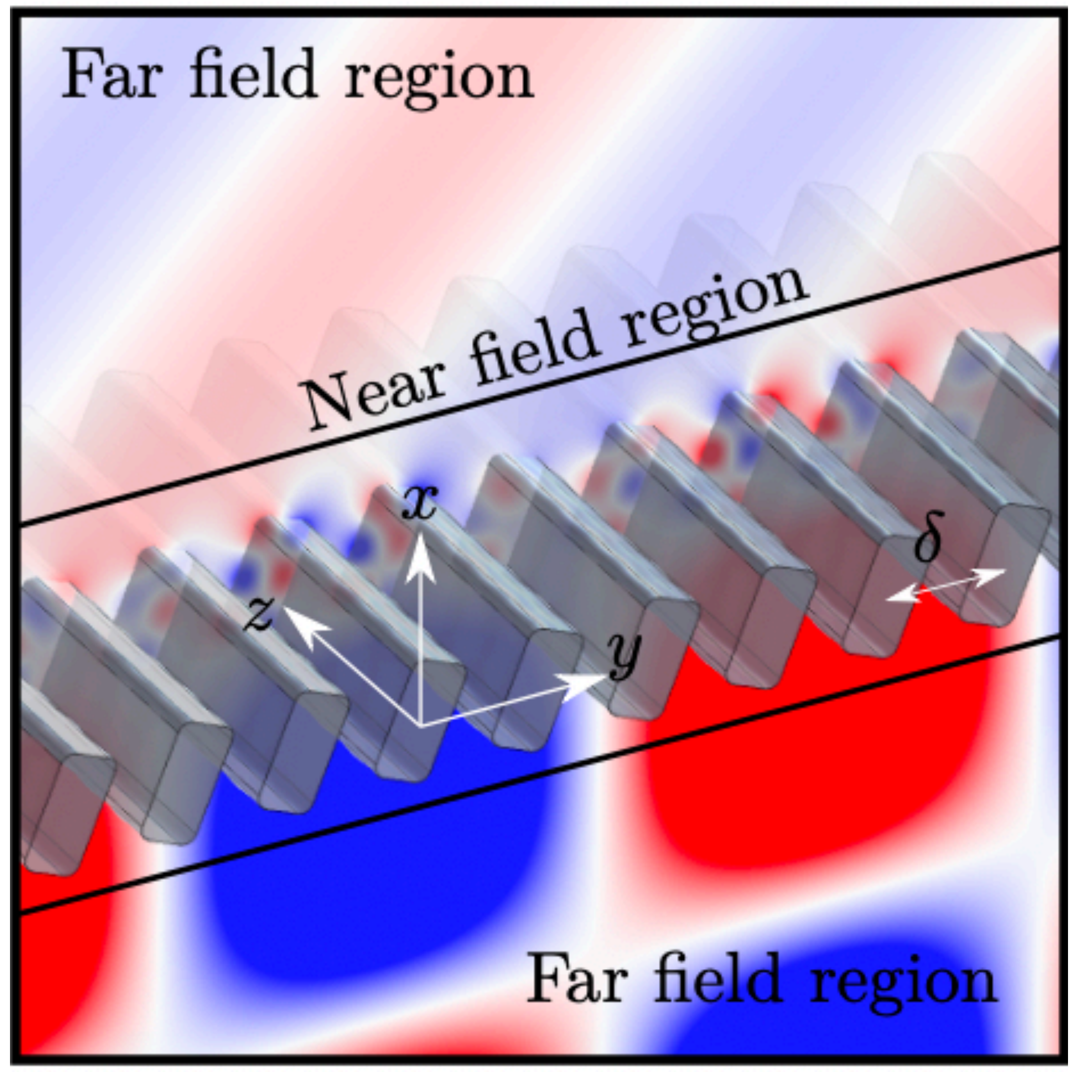
actual solution

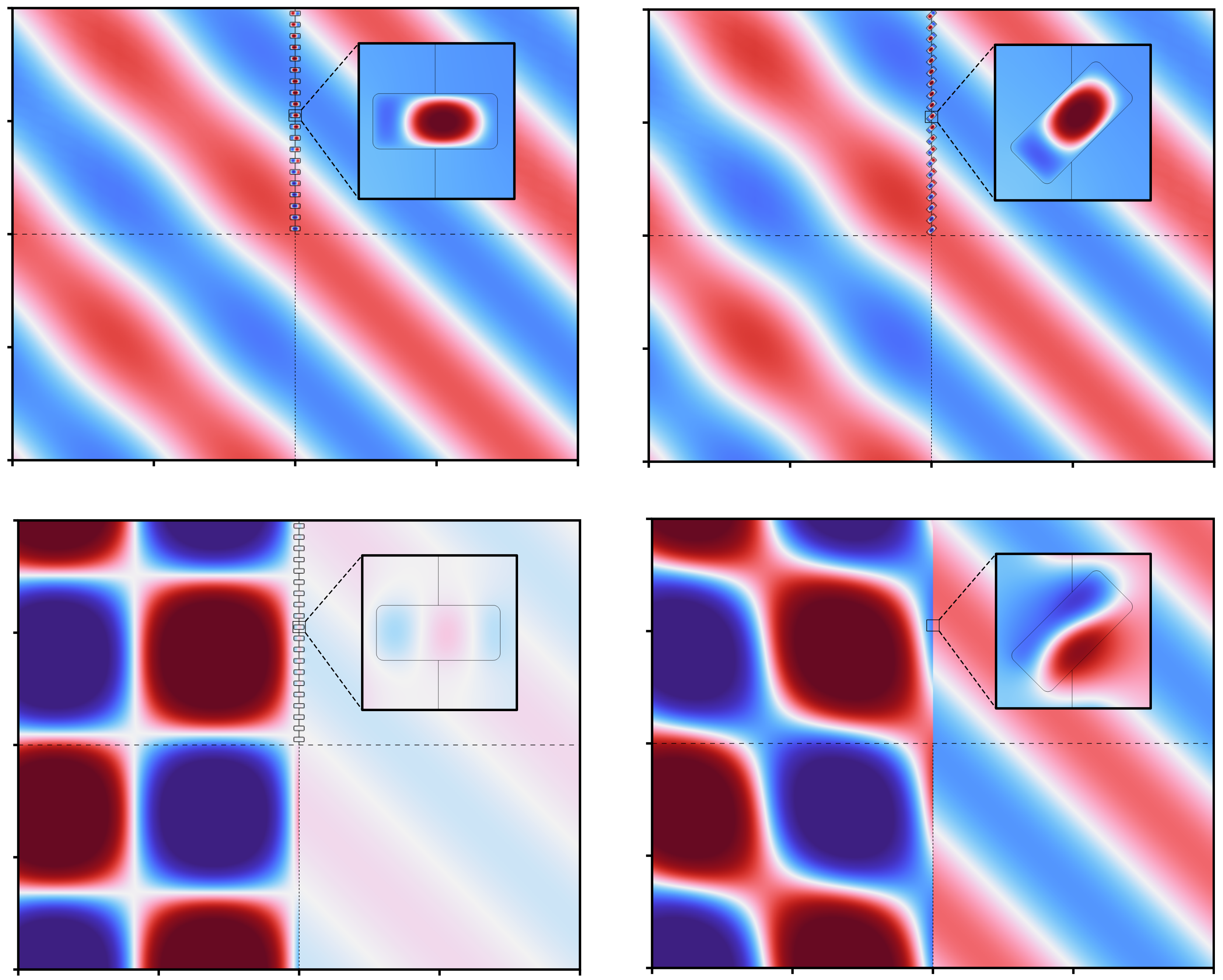
Homogenization

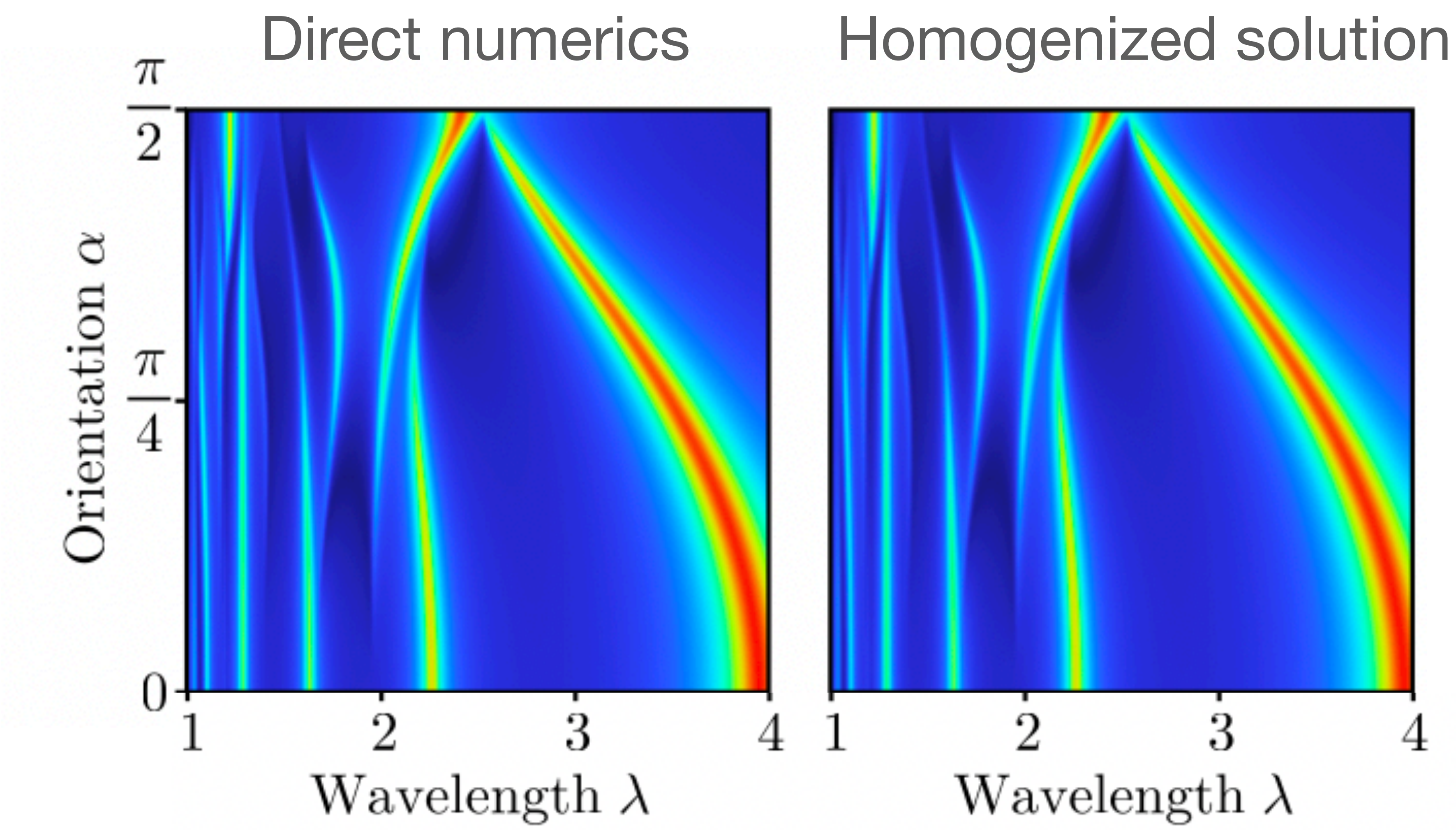
Examples in resonant cases

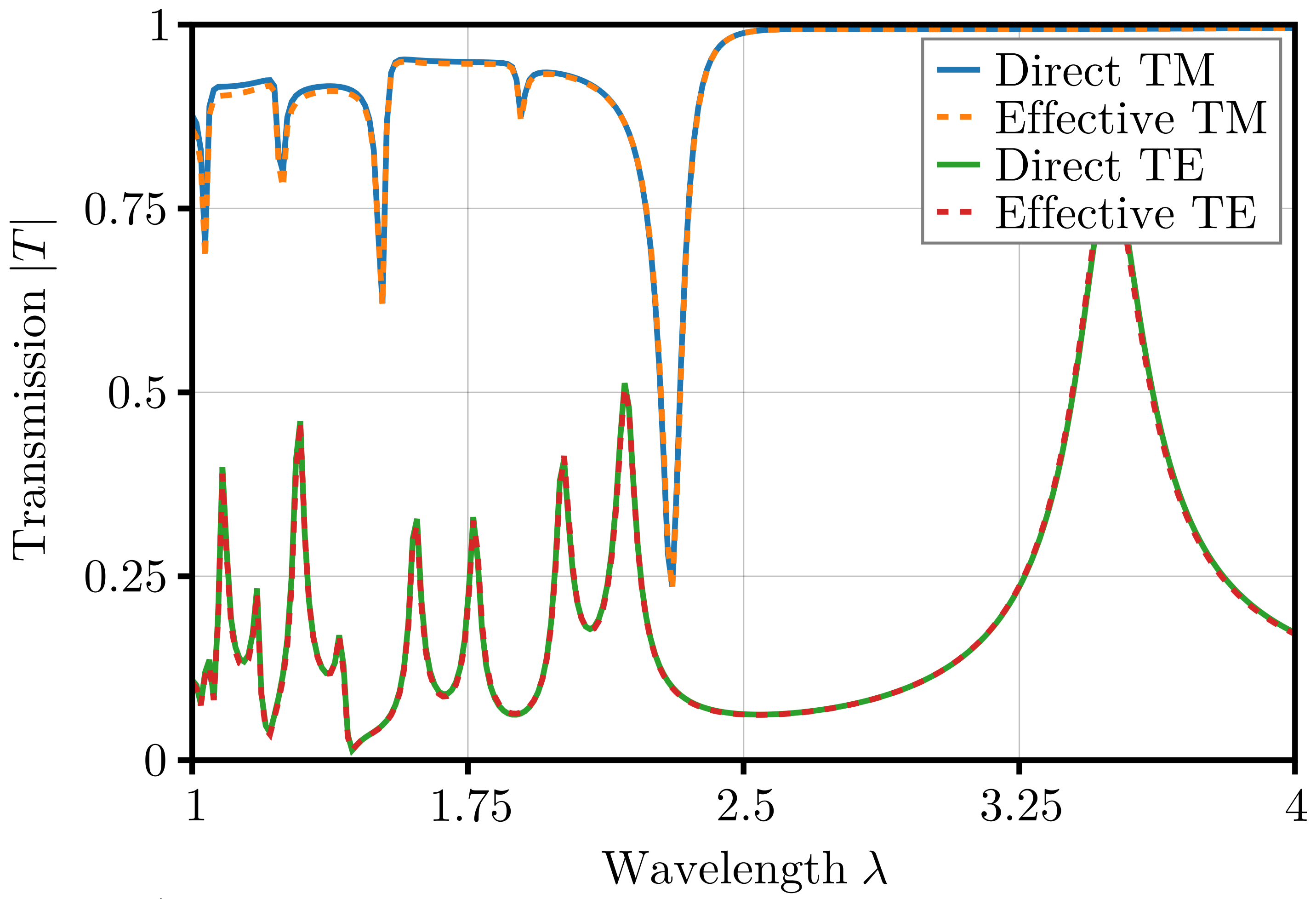
- Helmholtz resonance: B. Schweizer & A. Lamarcz, A. Maurel et al., O. Schnitzer, D. Abrahams
- Mie resonance : Felbacq & al., C. Boutin & Auriault, K. Pham et al.
- Minnaert resonance: K. Pham et al., H. Ammari et al.,
- Plasmonic resonance: N. Lebbe et al., H. Ammari, O. Schnitzer,
- Elastic longitudinal/flexural resonance: J.J. Marigo & al., A. Maurel & al.,

Mie resonance

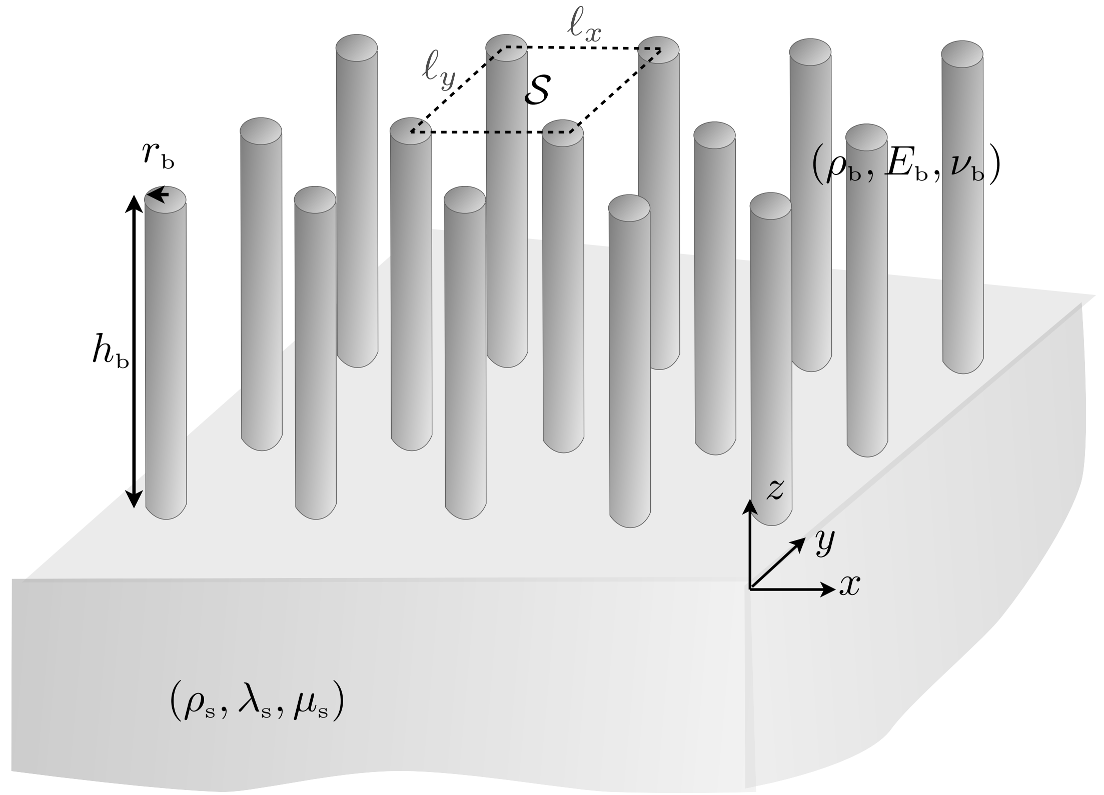


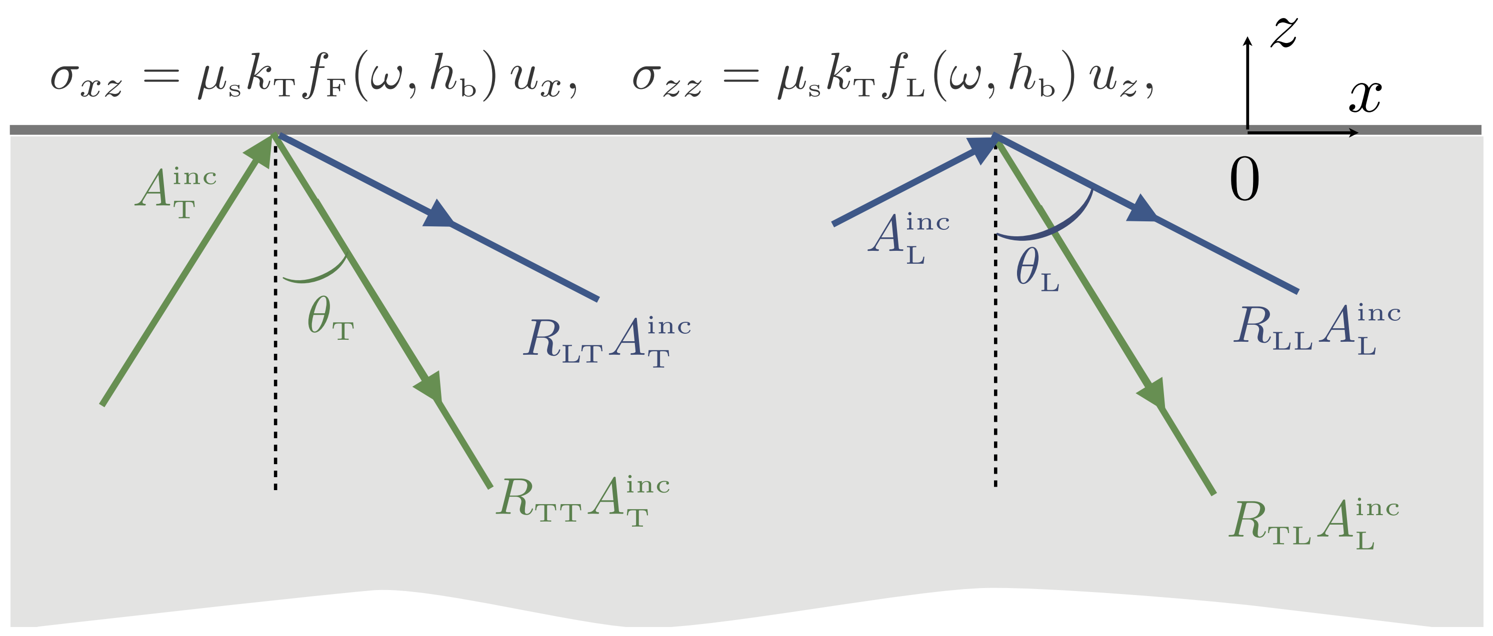




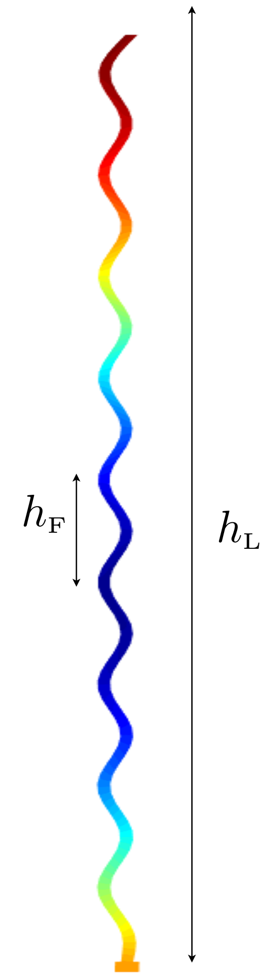


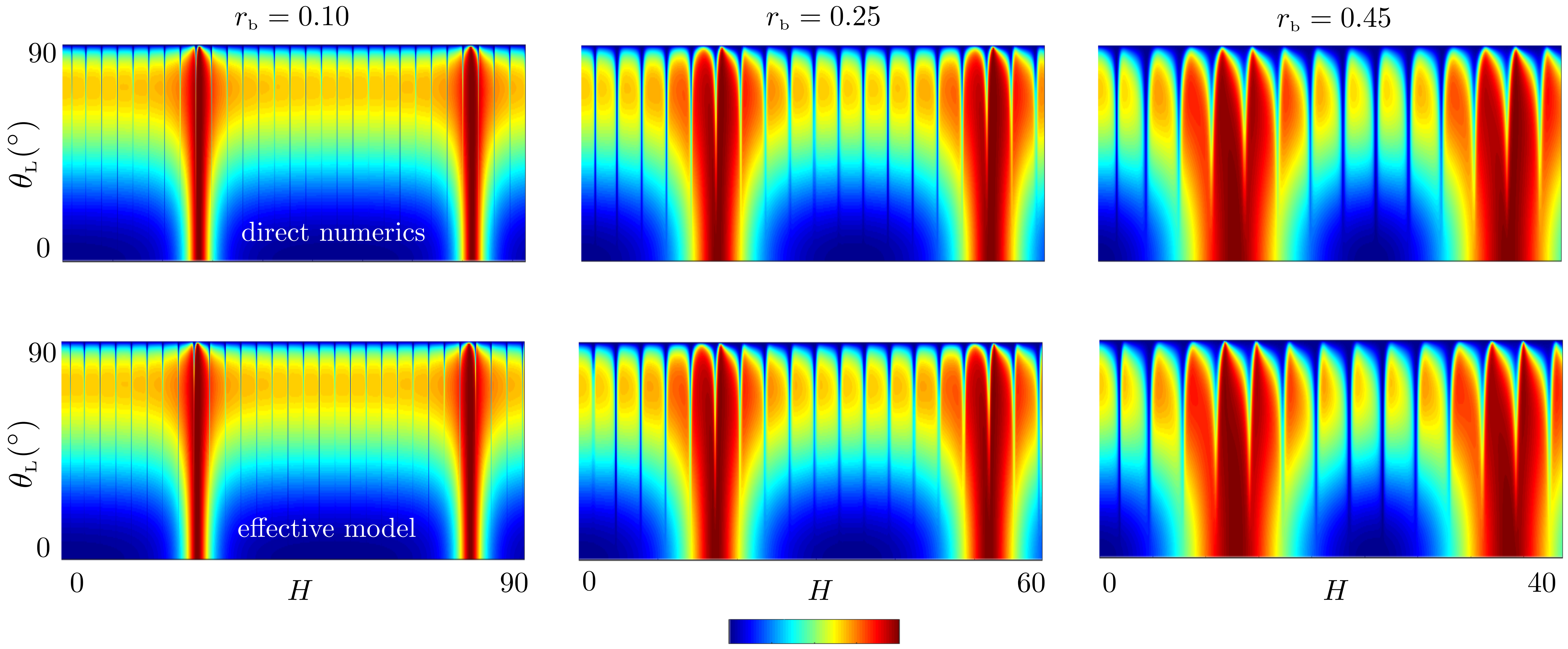
Flexural/longitudinal resonances of beam arrays

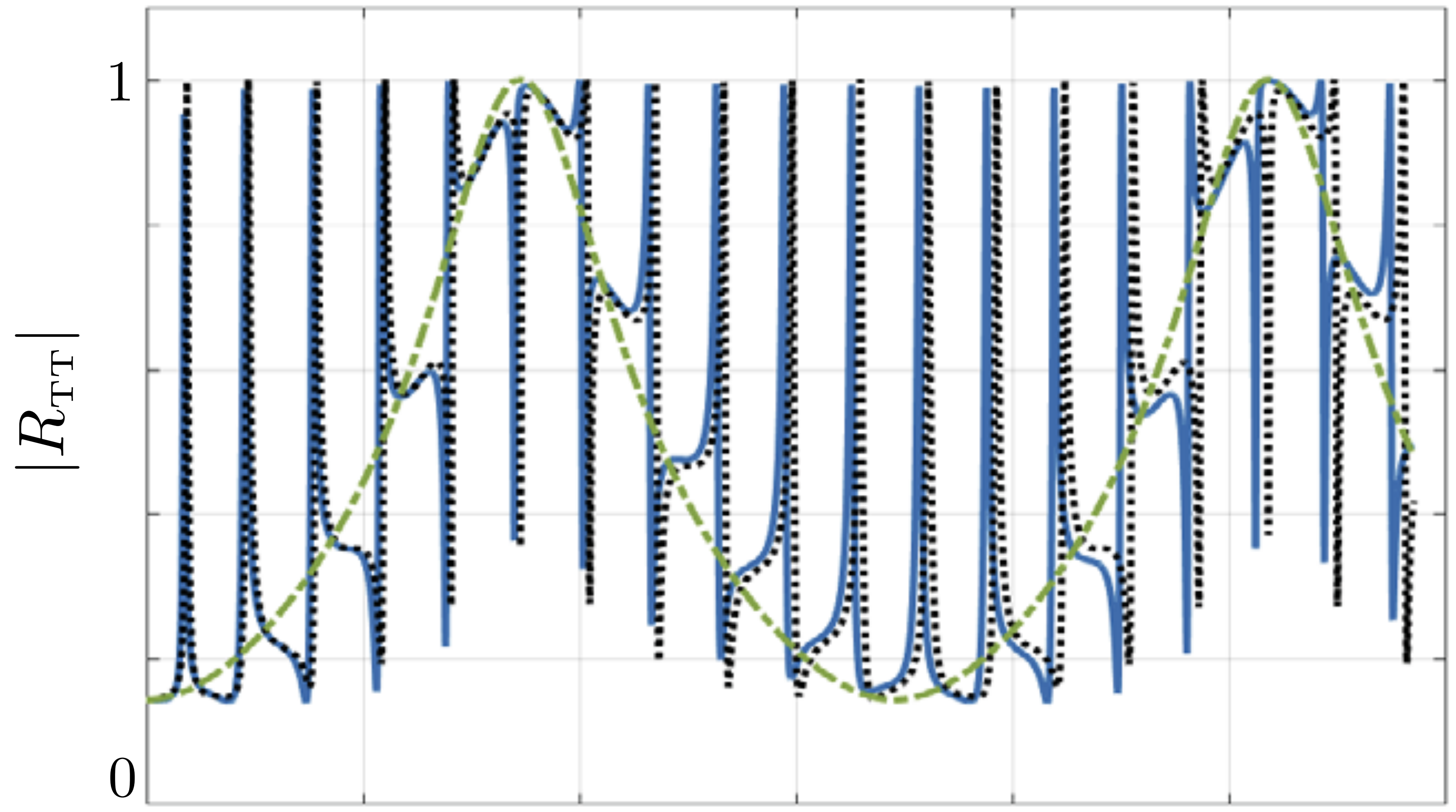




effective problem







Direct numerics

Homogenized solution