Introduction to Multiple Scattering

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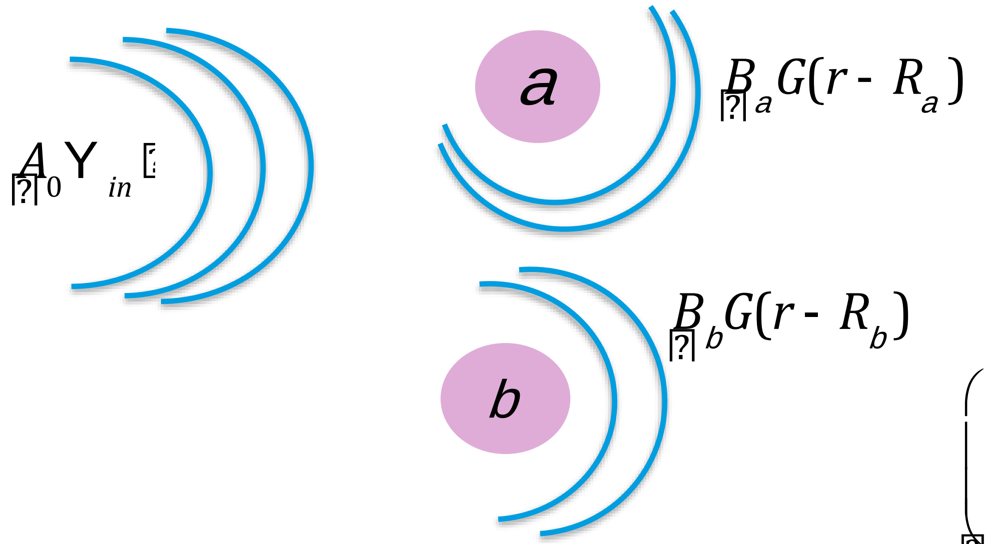
Multiple scattering



The total field is given by

$$\Psi = A_0\Psi_{in} + B_a G(r-R_a) + B_b G(r-R_b)$$

Where

$$\begin{pmatrix} \mu_a^{-1} & -G_{ab} \\ -G_{ba} & \mu_b^{-1} \end{pmatrix} \begin{pmatrix} B_a \\ B_b \end{pmatrix} = \begin{pmatrix} A_0\Psi_{in}(R_a) \\ A_0\Psi_{in}(R_b) \end{pmatrix}$$

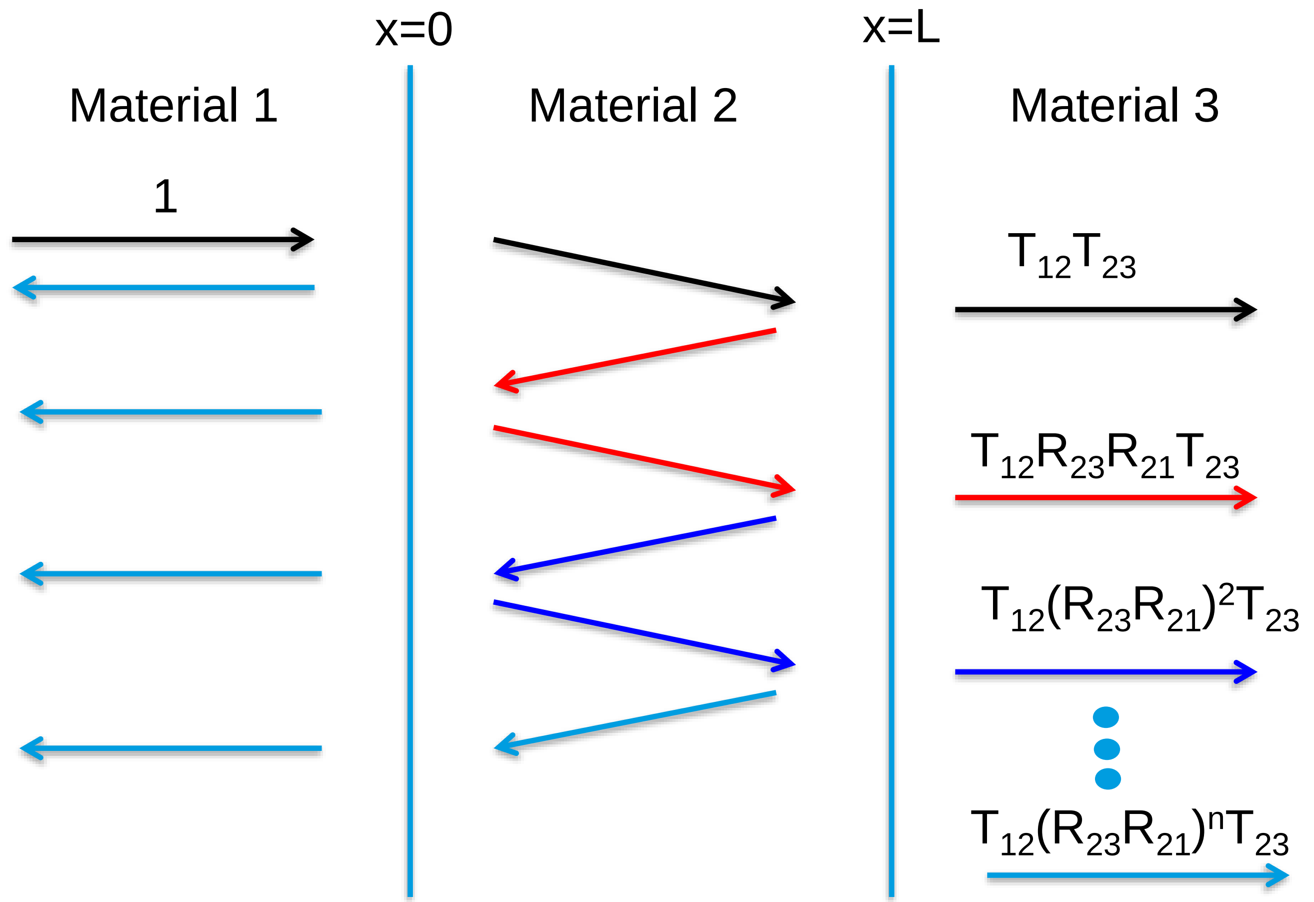
- The **μ** coefficients defines the scatterer
- Once known μ_α and μ_β , we solve for B_α and B_β

Multiple Scattering Theory

Outline:

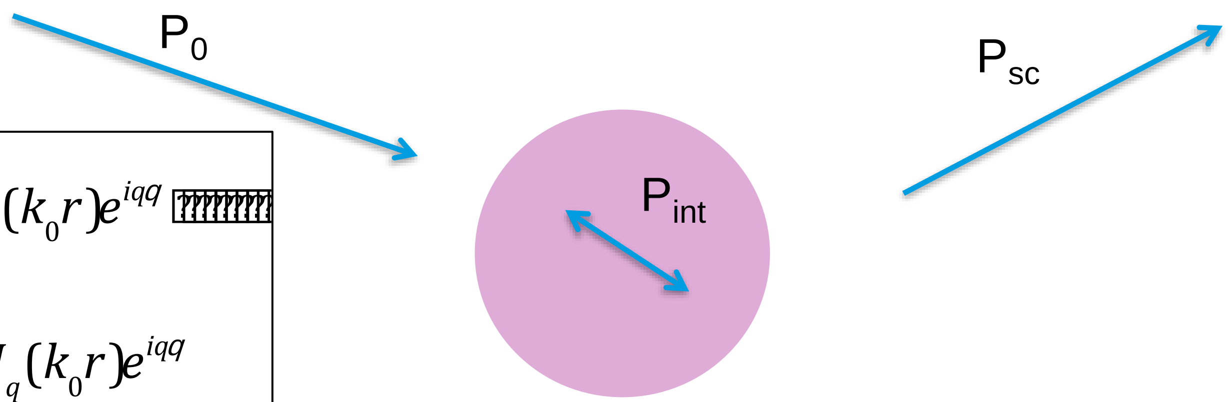
1. Introduction to MST
2. Explicit expressions for 2D
3. T-Matrix of a cluster
4. Homogenization
5. Flexural waves and point scatterers

Reflection and Transmission by a Slab



$$T = T_{12}T_{23}\sum_{n=0}^{\infty}(R_{21}R_{23})^n = \frac{T_{12}T_{23}}{1 - R_{21}R_{23}}$$

Scattering by Circular Cylindrical Objects



$$P_0 = \sum_q A_q J_q(k_0 r) e^{iq\theta}$$

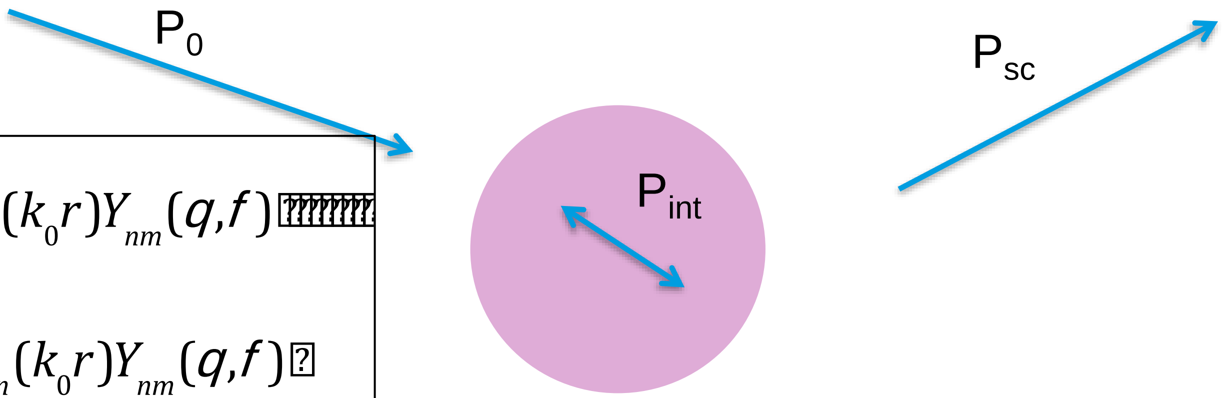
$$P_{sc} = \sum_q B_q H_q(k_0 r) e^{iq\theta}$$

$$P_{\text{int}} = \sum_q C_q J_q(k_a r) e^{iq\theta}$$

$$\sum_q A_q J_q(k_0 R_a) e^{iq\theta} + \sum_q B_q H_q(k_0 R_a) e^{iq\theta} = \sum_q C_q J_q(k_a R_a) e^{iq\theta}$$

$$Z_0 \left(\sum_q A_q \partial_r J_q(k_0 R_a) e^{iq\theta} + \sum_q B_q \partial_r H_q(k_0 R_a) e^{iq\theta} \right) = Z_a \sum_q C_q \partial_r J_q(k_a R_a) e^{iq\theta}$$

Scattering by Spherical Objects



$$P_0 = \sum_n A_n j_n(k_0 r) Y_{nm}(\theta,\phi)$$

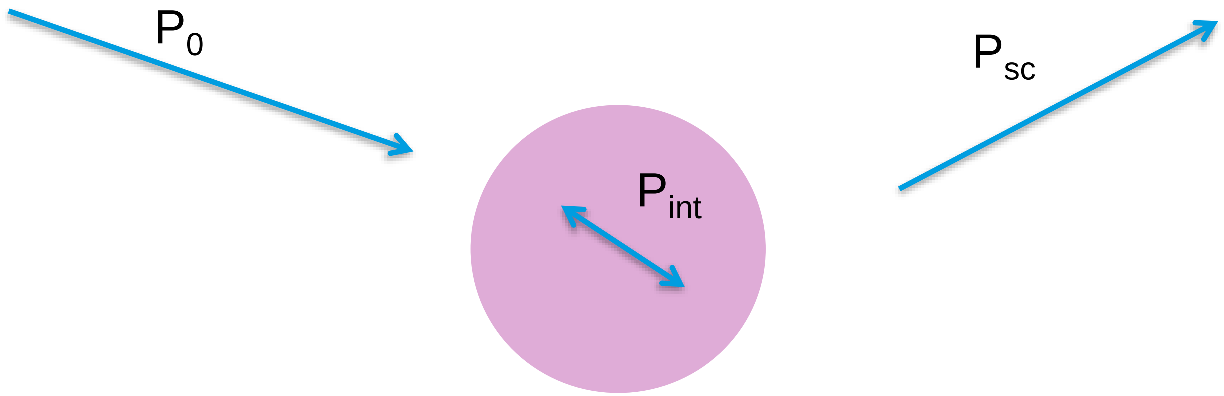
$$P_{sc} = \sum_n B_n h_n(k_0 r) Y_{nm}(\theta,\phi)$$

$$P_{\text{int}} = \sum_n C_n j_n(k_a r) Y_{nm}(\theta,\phi)$$

$$\sum_{n,m} A_n j_n(k_0 r) Y_{nm}(\theta,\phi) + \sum_{n,m} B_n h_n(k_0 r) Y_{nm}(\theta,\phi) = \sum_{n,m} C_n j_n(k_a r) Y_{nm}(\theta,\phi)$$

$$Z_0 \left(\sum_{n,m} A_n \partial_r j_n(k_0 r) Y_{nm}(\theta,\phi) + \sum_{n,m} B_n \partial_r h_n(k_0 r) Y_{nm}(\theta,\phi) \right) = Z_a \sum_{n,m} C_n \partial_r j_n(k_a r) Y_{nm}(\theta,\phi)$$

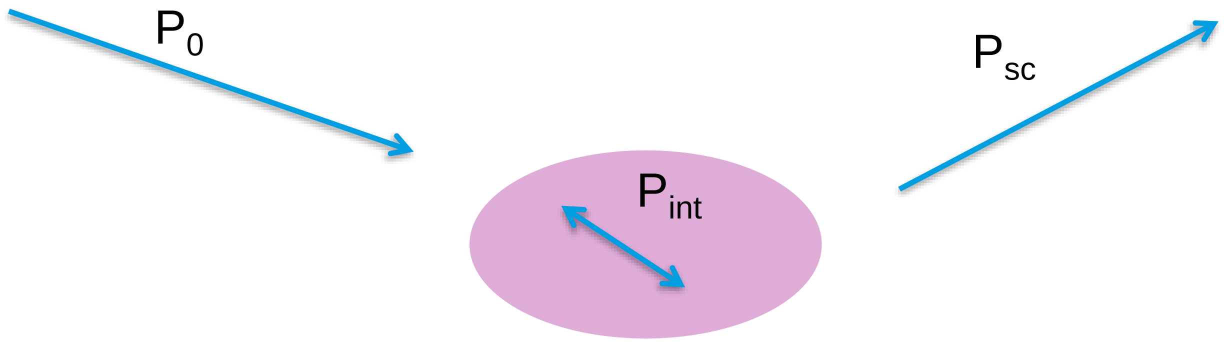
Scattering by Cylindrical and Spherical Objects



$$B_q = R_q A_q \rightarrow R_q \equiv \text{T?Matrix}$$

$$C_q = T_q A_q$$

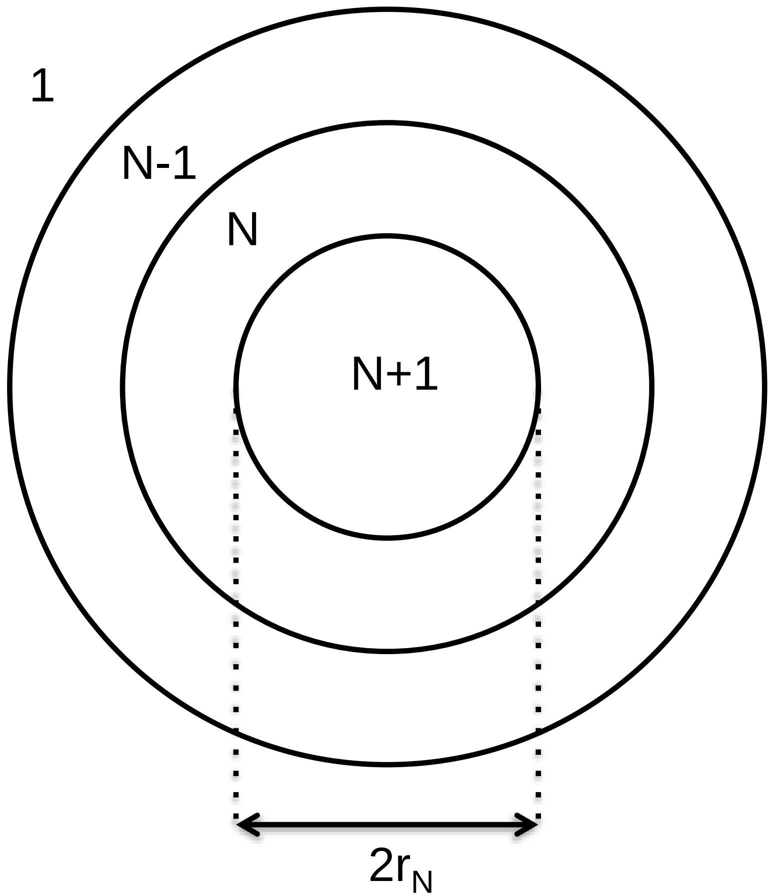
Scattering by Cylindrical and Spherical Objects



$$B_q = \sum_s R_{qs} A_s$$

$$C_q = \sum_s T_{qs} A_s$$

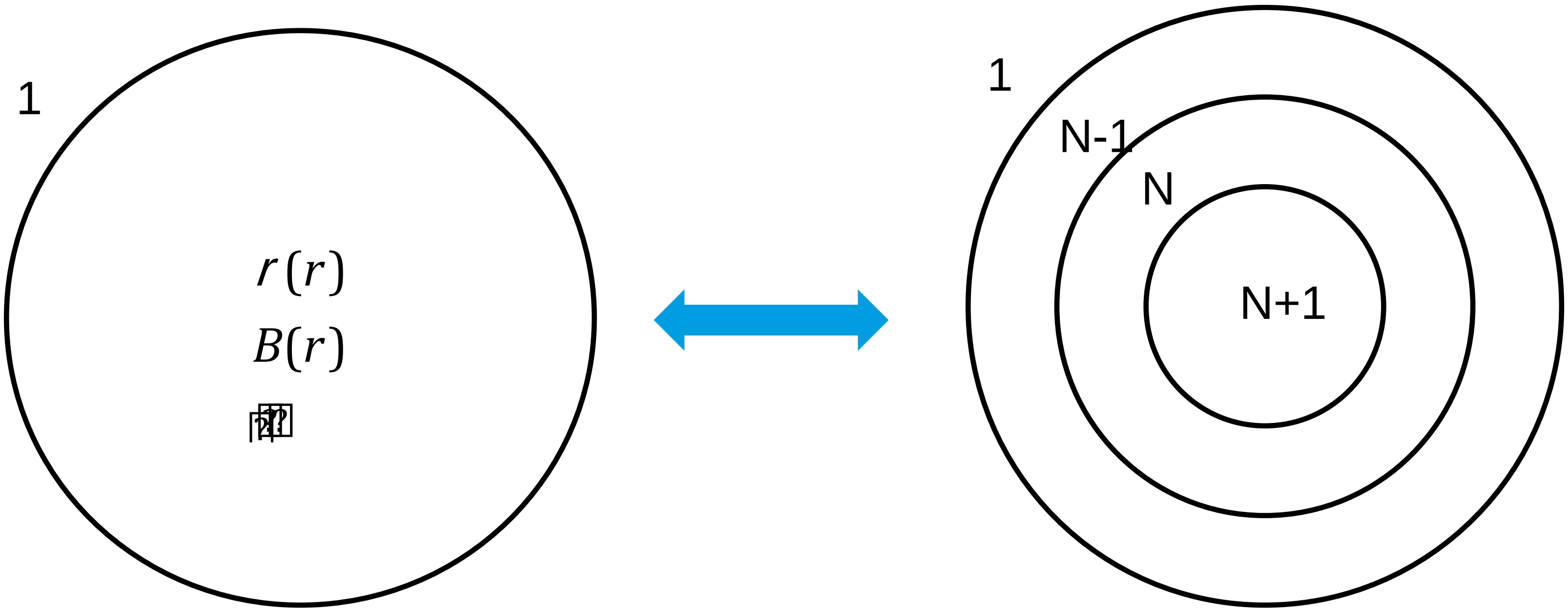
Scattering by Layered Objects



N layers
N+2 materials
N+1 interfaces: Material n is located between $r=r_{n-1}$ and $r=r_n$

Scattering by Layered Objects

Scattering of inhomogeneous objects by discretization

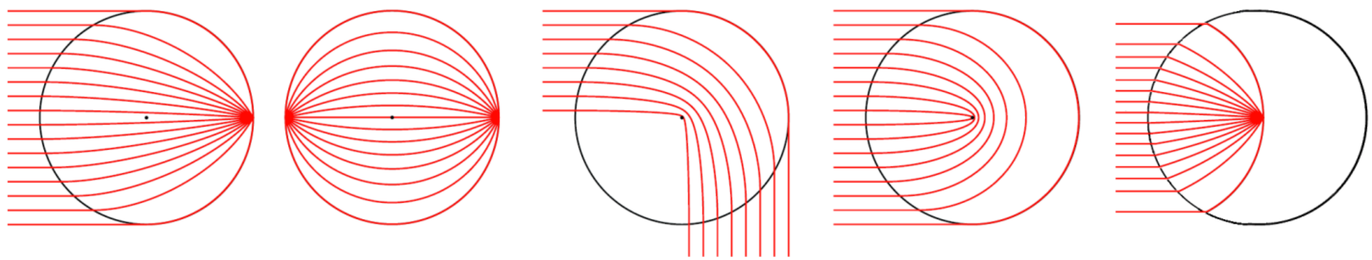


$$r(r) = r_N(r_N)$$

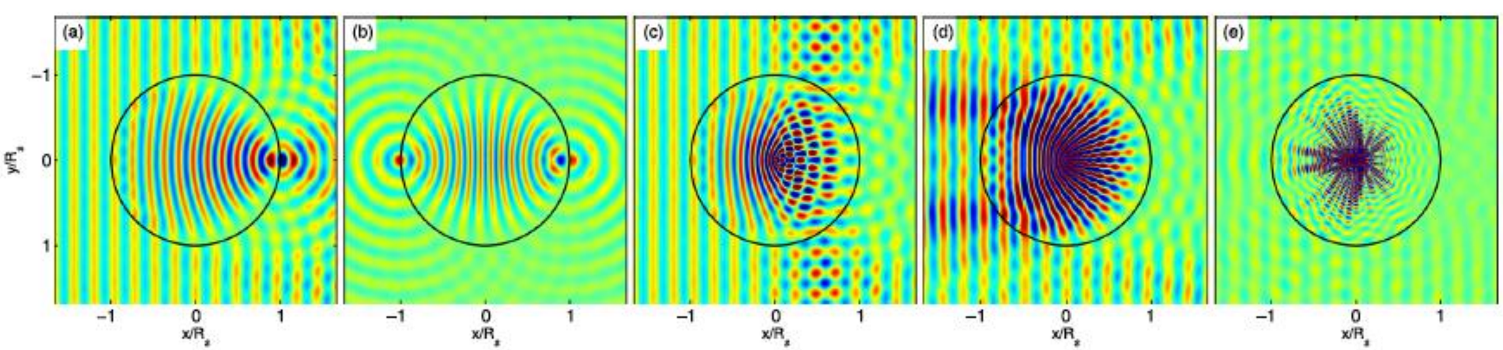
$$B(r) = B_N(r_N)$$

Design and Simulation of Omnidirectional Lenses

Luneburg	Maxwell	90º Rotating	Eaton	Concentrator



Lens Name	Refractive Index (n)
Luneburg	$n = \sqrt{2 - r^2}$
Maxwell Fish-Eye	$n = 2/(1 + r^2)$
90° Rotating	$rn^4 - 2n + r = 0$
Eaton	$n = \sqrt{2/r - 1}$
Concentrator	$n = 1/r$



Change of Notation!!

$$P_0 = \sum_q A_q J_q(k_0 r) e^{iq\theta}$$

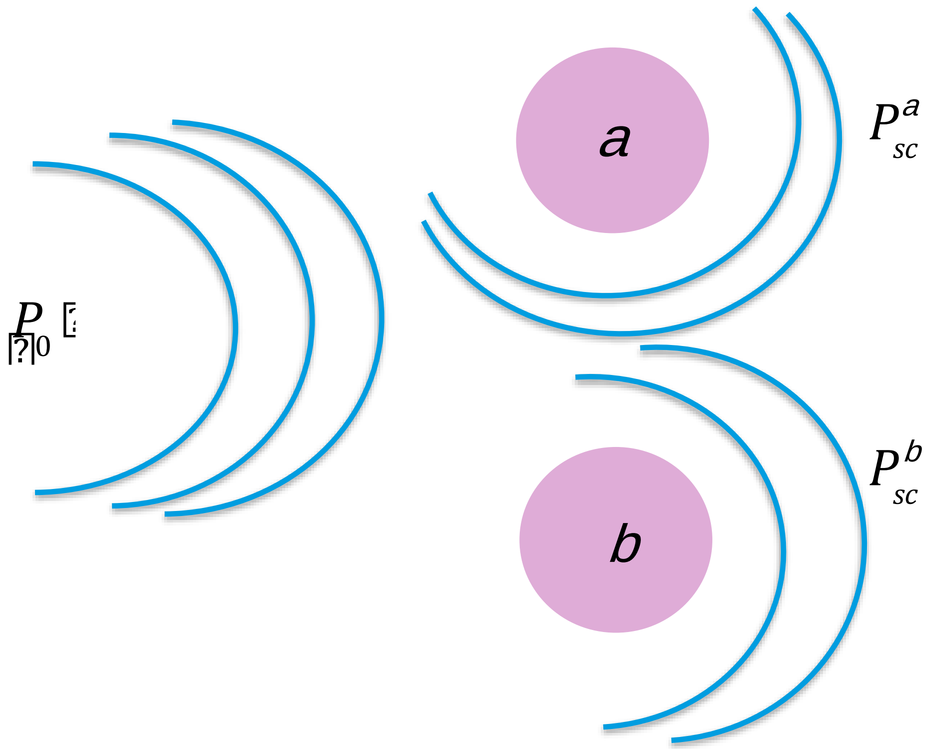
$$P_{sc} = \sum_q B_q H_q(k_0 r) e^{iq\theta}$$

$$P_{\text{int}} = \sum_q C_q J_q(k_a r) e^{iq\theta}$$

$$B = TA$$

Now T will be the T-Matrix, and relates the incident field A with the scattered field B

In multiple scattering we don’t care about the field inside the scatterer



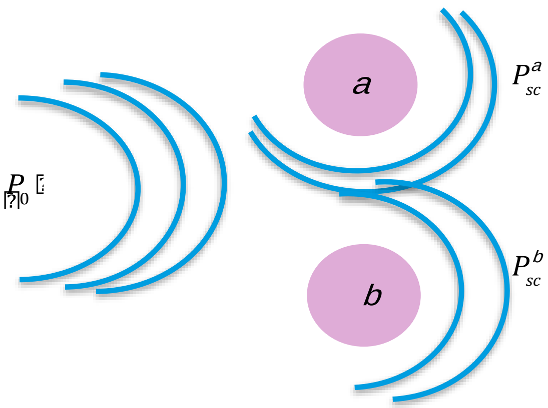
$$P = P_0 + P_{sc}^a + P_{sc}^b$$

$$P_{sc}^a = T_a P_0^a = T_a \left(P_0 + G_{ab} P_{sc}^b \right)$$

$$P_{sc}^b = T_b P_0^b = T_b \left(P_0 + G_{ba} P_{sc}^a \right)$$

$$\begin{pmatrix} 1 & -T_a G_{ab} \\ -T_b G_{ba} & 1 \end{pmatrix} \begin{pmatrix} P_{sc}^a \\ P_{sc}^b \end{pmatrix} = \begin{pmatrix} T_a \\ T_b \end{pmatrix} P_0$$

Introduction to Multiple Scattering



$$\begin{pmatrix} 1 & -T_a G_{ab} \\ -T_b G_{ba} & 1 \end{pmatrix} \begin{pmatrix} P^a_{sc} \\ P^b_{sc} \end{pmatrix} = \begin{pmatrix} T_a \\ T_b \end{pmatrix} P_0$$

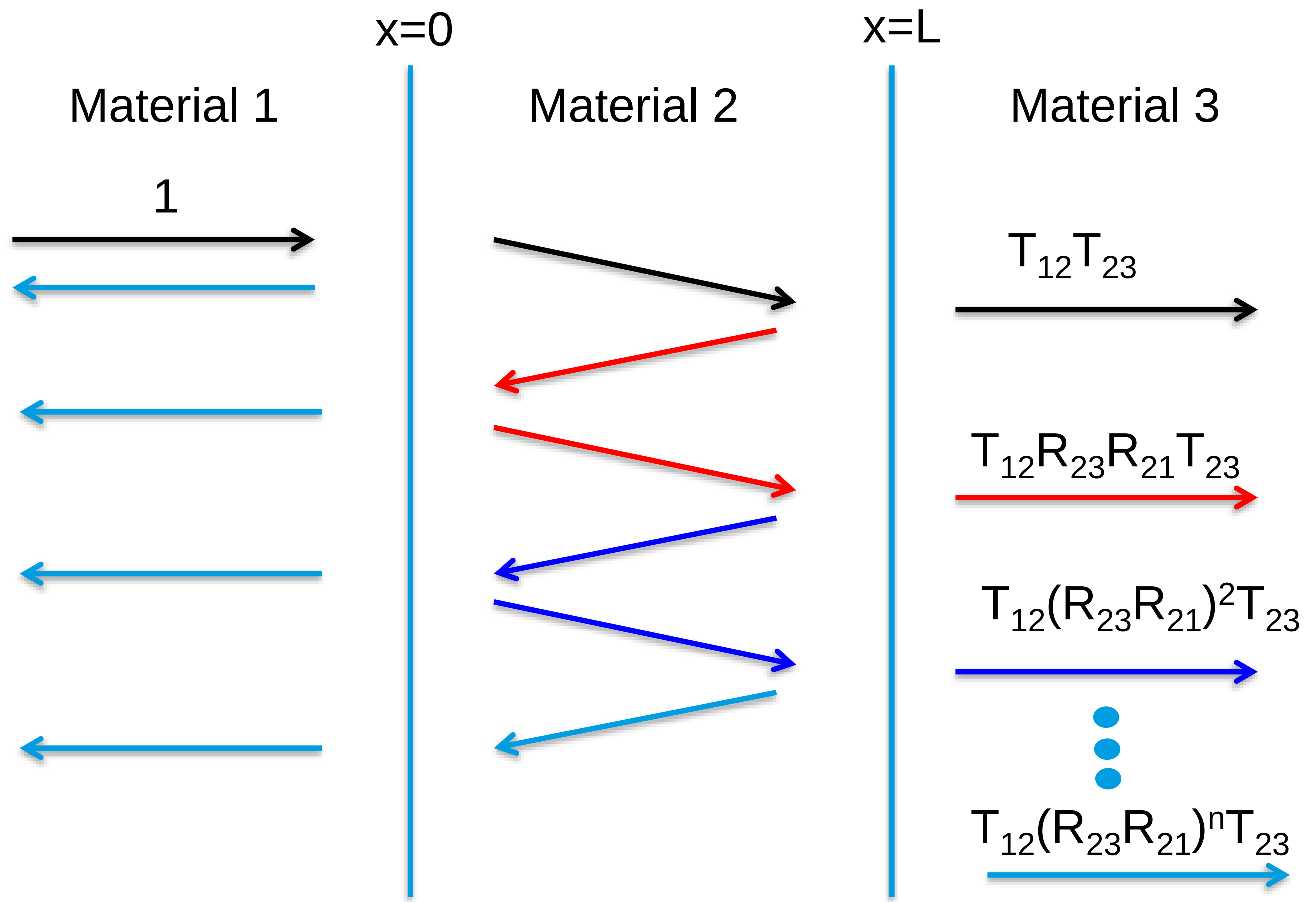
$$M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & -T_a G_{ab} \\ -T_b G_{ba} & 0 \end{pmatrix}$$

$$MP_{sc} = TP_0 \rightarrow (I - TG)P_{sc} = TP_0 \rightarrow P_{sc} = (I - TG)^{-1} TP$$

$$(I - TG)^{-1} \approx I + TG + (TG)^2 + (TG)^3 + \cdots$$

$$P_{sc} \approx TP_0 + TGTP_0 + (TG)^2 TP_0 + (TG)^3 TP_0 + \cdots$$

Reflection and Transmission by a Slab



$$T = T_{12}T_{23}\sum_{n=0}^{\infty}(R_{21}R_{23})^n = \frac{T_{12}T_{23}}{1 - R_{21}R_{23}}$$

Introduction to Multiple Scattering

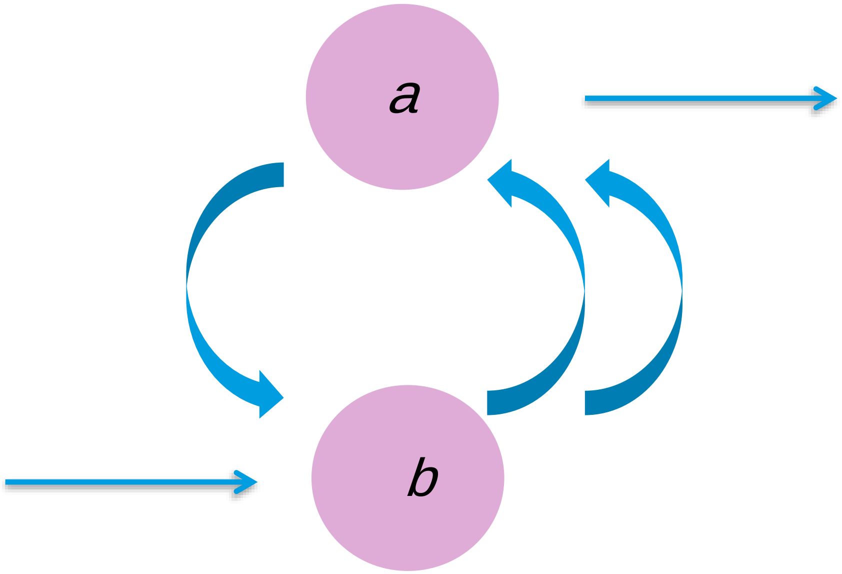
$$P_{sc} \approx TP_0 + TGTP_0 + (TG)^2 TP_0 + (TG)^3 TP_0 + \cdots$$

TP_0

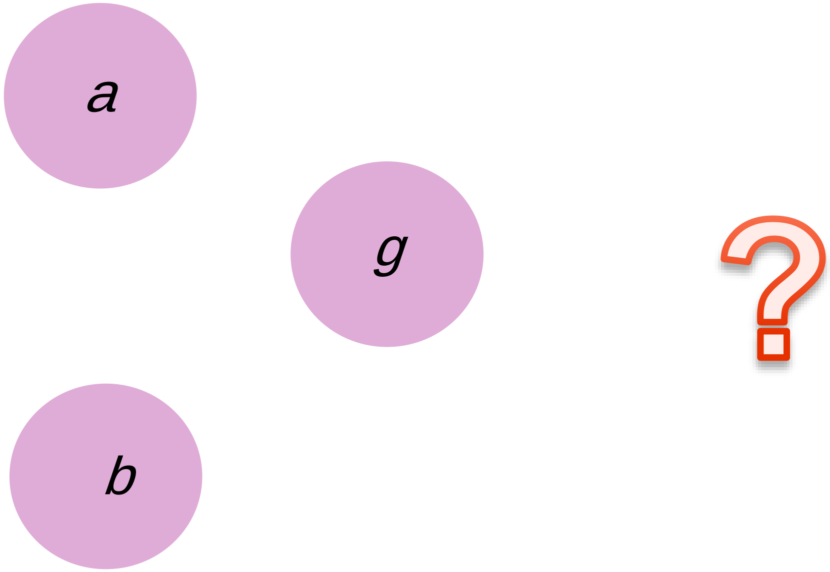
$TGTP_0$

$(TG)^2 TP_0$

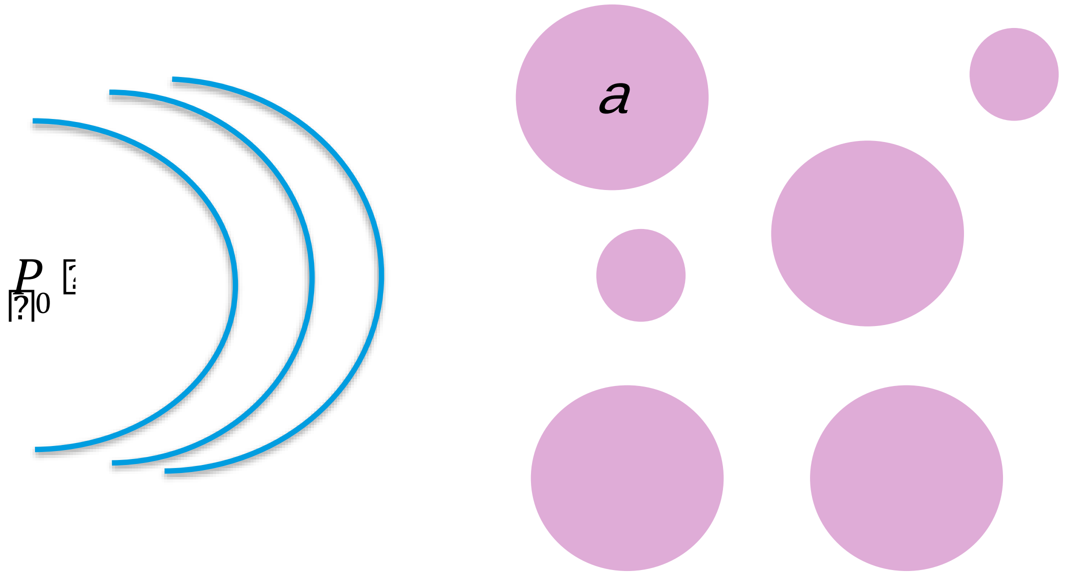
$(TG)^3 TP_0$



Introduction to Multiple Scattering



Introduction to Multiple Scattering



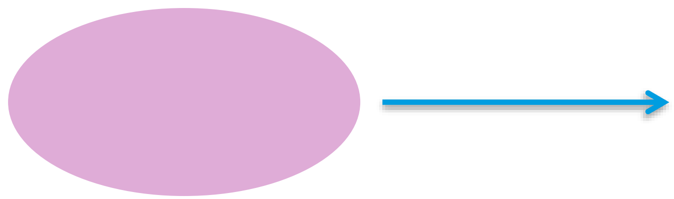
$$P_{sc}^{a} = T_a P_0 + T_a \sum_{b \neq a}^{N} G_{ab} P_{sc}^{b} \rightarrow M_{ab} = \delta_{ab} - T_a G_{ab}$$

$$\sum_{q,b} \left(\delta_{ba} \delta_{qs} - T_s^a G_{ba}^{qs} \right) B_q^b = T_s^a A_s^{0a} \rightarrow (2Q_{\max} + 1) * N \text{ Eqs. and Unknowns}$$

Multiple Scattering by General Cylindrical Objects

$$\sum_{q,\beta}\left(\delta_{\beta\alpha}\delta_{qs} - T_s^{\alpha} G_{\beta\alpha}^{qs}\right) B_q^{\beta} = T_s^{\alpha} A_s^{0\alpha} \rightarrow \sum_{q,\beta} M_{sq}^{\alpha\beta} B_q^{\beta} = T_s^{\alpha} A_s^{0\alpha}$$

$$M_{sq}^{\alpha\beta} = \delta_{\beta\alpha}\delta_{qs} - T_s^{\alpha} G_{\beta\alpha}^{qs}$$

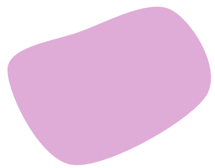


$$T_{sp}^{\alpha}$$

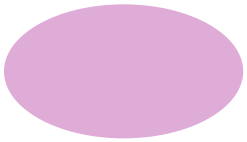
$$M_{sq}^{\alpha\beta} = \delta_{\beta\alpha}\delta_{qs} - \sum_{p} T_{sp}^{\alpha} G_{\beta\alpha}^{qp}$$

Waterman, P. C. (1969). New formulation of acoustic scattering. The journal of the acoustical society of America, 45(6), 1417-1429.

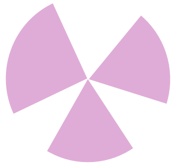
Multiple Scattering by General Cylindrical Objects



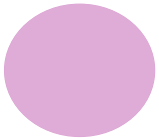
Waterman T-matrix



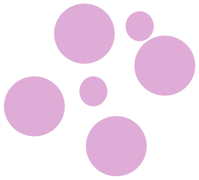
Mathieu Functions
(Bessel functions in elliptical coordinates)



Mode matching for rigid sectors



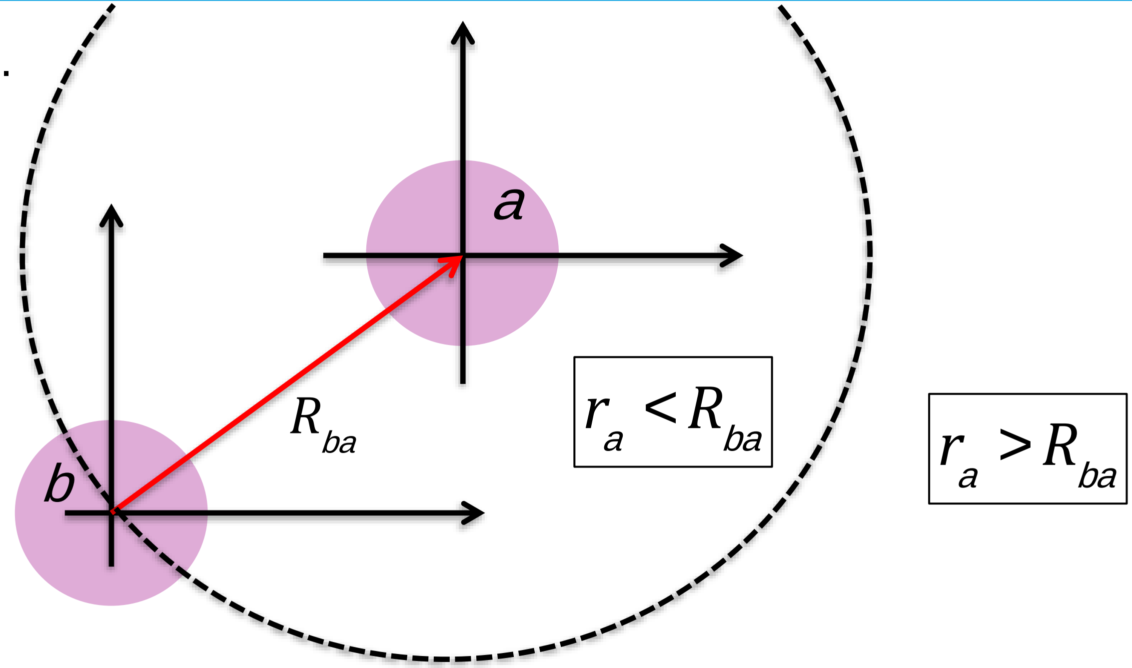
Anisotropic Metafluids
(Integral methods)



Addition theorem and MST

Addition Theorem in 2D

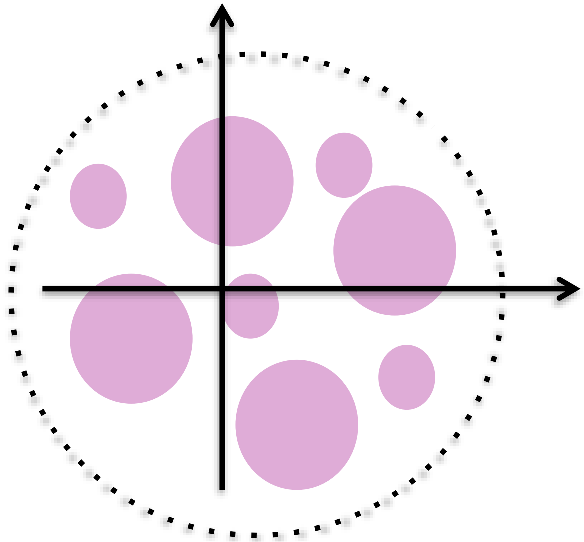
Abramowitz, M., & Stegun, I. A.



$$r_a < R_{ba} \rightarrow H_q(kr_b)e^{iqJ_b} = \sum_s H_{q-s}(kR_{ba})e^{i(q-s)J_{ba}} J_s(kr_a)e^{isJ_a}$$

$$r_a > R_{ba} \rightarrow H_q(kr_b)e^{iqJ_b} = \sum_s J_{q-s}(kR_{ba})e^{i(q-s)J_{ba}} H_s(kr_a)e^{isJ_a}$$

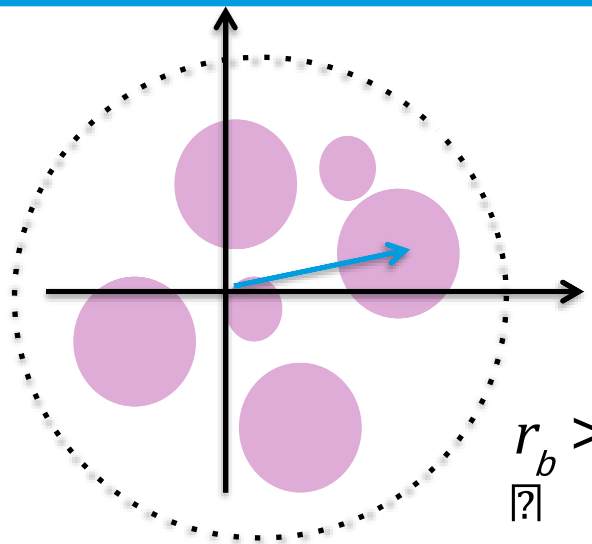
T-Matrix of a Cluster



$$P_{sc} = \sum_b P^b_{sc} = \sum_b \sum_q B^b_q H_q(kr_b) e^{iqJ_b} \longrightarrow P_{sc} = \sum_s B^{cls}_s H_s(kr) e^{isJ}$$

$$\sum_{b,q} M^{ab}_{sq} B^b_q = T^a_s A_s \rightarrow B^b_q = \sum_{a,s} (M^{ba}_{qs})^{-1} T^a_s A_s \longrightarrow \boxed{B^{cls}_s = \sum_q T^{cls}_{sq} A_s}$$

T-Matrix of a Cluster



$$P_{sc} = \sum_b P_{sc}^b = \sum_b \sum_q B_q^b H_q(kr_b) e^{iqJ_b}$$

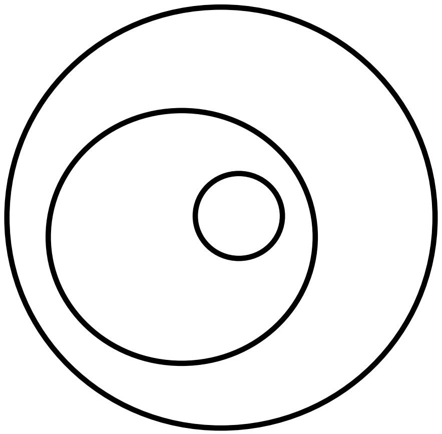
$$r_b > R_{b0} \rightarrow H_q(kr_b) e^{iqJ_b} = \sum_s J_{q-s}(kR_b) e^{i(q-s)J_b} H_s(kr) e^{isJ}$$

$$P_{sc} = \sum_b \sum_q \sum_s J_{q-s}(kR_b) e^{i(q-s)J_b} B_q^b H_s(kr) e^{isJ} = \sum_s B_s^{cls} H_s(kr) e^{isJ}$$

$$B_s^{cls} = \sum_b \sum_q \sum_s J_{q-s}(kR_b) e^{i(q-s)J_b} B_q^b = \sum_b \sum_q \sum_{s,p} J_{q-s}(kR_b) e^{i(q-s)J_b} (M_{qp}^{ba})^{-1} T_p^a A_p$$

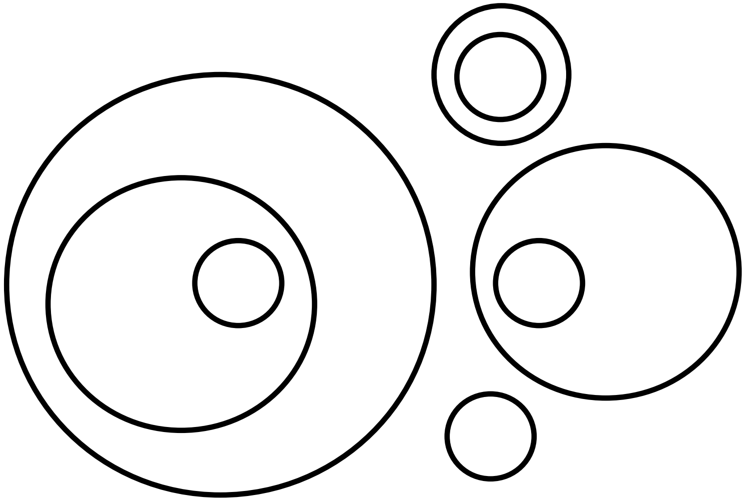
$$B_s^{cls} = \sum_p T_{sp}^{cls} A_p \rightarrow T_{sp}^{cls} = \sum_b \sum_q \sum_{s,p} J_{q-s}(kR_b) e^{i(q-s)J_b} (M_{qp}^b)^{-1} T_p^b$$

T-Matrix of Non-Concentric Shells

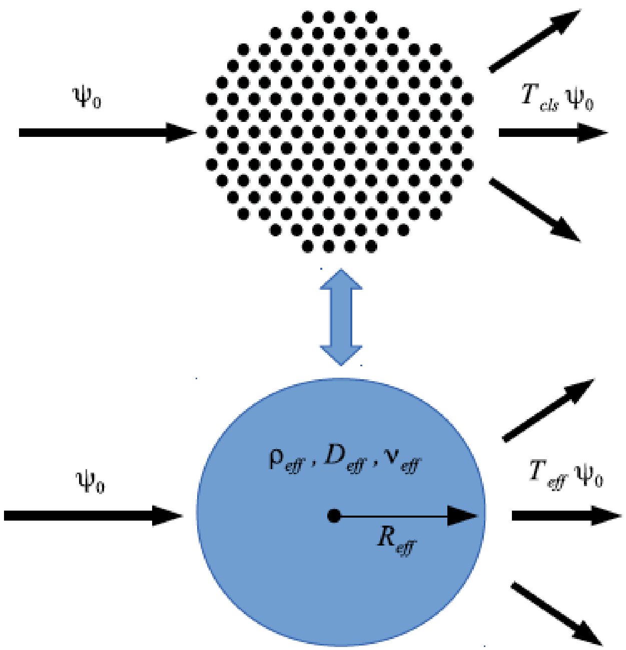


Addition Theorem + Layered Scatterers

Addition Theorem + Layered Scatterers+MST



Homogenization



$$\lim_{\omega\to 0} T^{cls} = \lim_{\omega\to 0} T^{eff}$$

For q=0,1

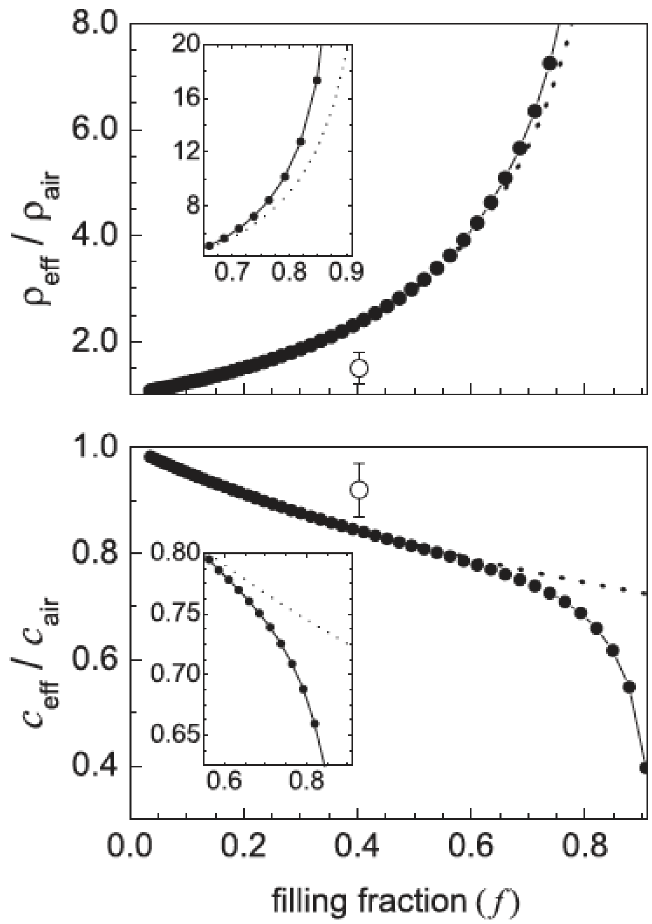
$$T_q^{eff} \approx \frac{i\pi k_b^2 R_{eff}^2}{4}\Gamma_q^{eff} \qquad T_q^{cls} \approx \frac{i\pi k_b^2 R_a^2}{4} N\Gamma_q^{cls}$$

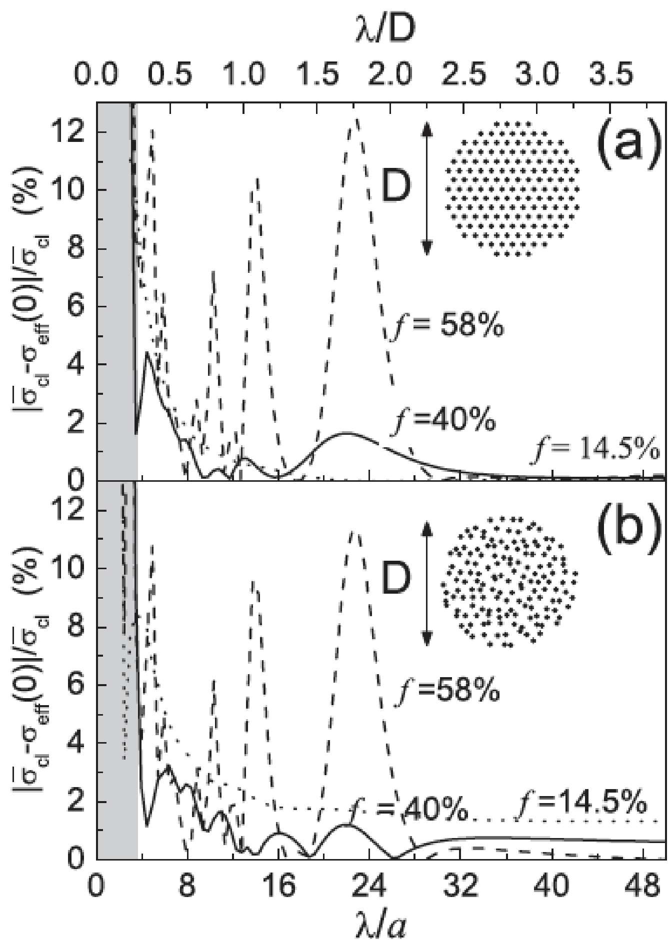
$$\Gamma_q^{eff} = f\Gamma_q^{cls} \qquad f \approx \frac{NR_a^2}{R_{eff}^2}$$

$$\left.\begin{aligned}\Gamma_0^{eff} &= 1 - B_b/B_{eff}\\ \Gamma_0^{cls} &= 1 - B_b/B_a\end{aligned}\right\} 1/B_{eff} = (1-f)/B_b + f/B_a$$

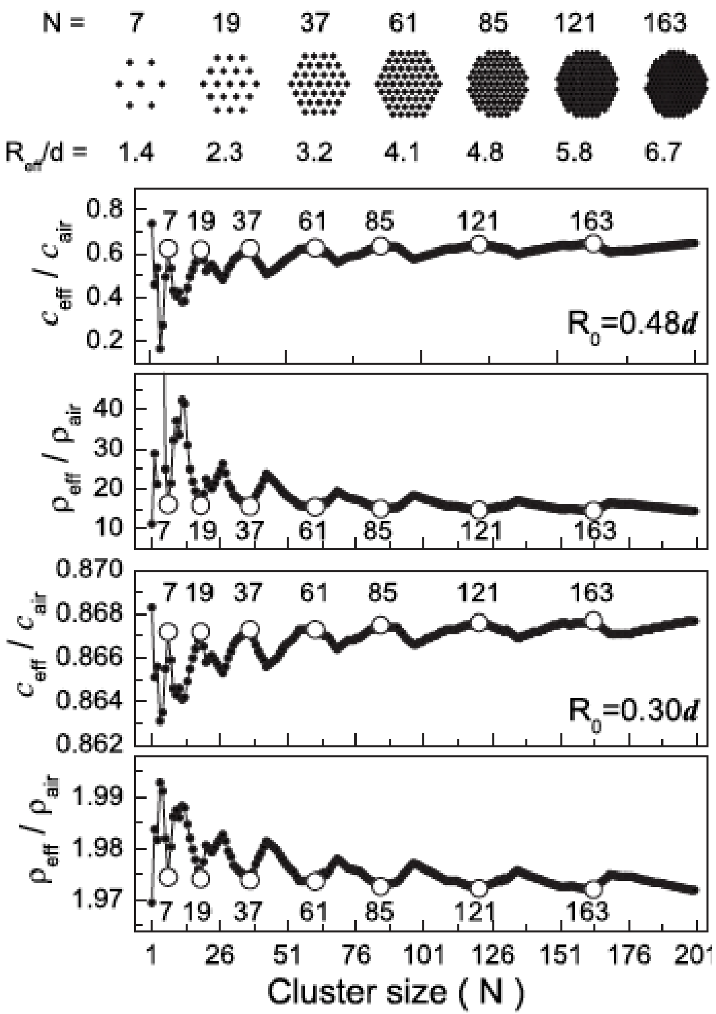
$$\left.\begin{aligned}\Gamma_1^{eff} &= \frac{\rho_{eff}-\rho_b}{\rho_{eff}+\rho_b}\\ \Gamma_1^{cls} &= \Delta\frac{\rho_a-\rho_b}{\rho_a+\rho_b}\end{aligned}\right\} \rho_{eff}/\rho_b = \frac{\rho_a(\Delta+f)+\rho_b(\Delta-f)}{\rho_a(\Delta-f)+\rho_b(\Delta+f)}$$

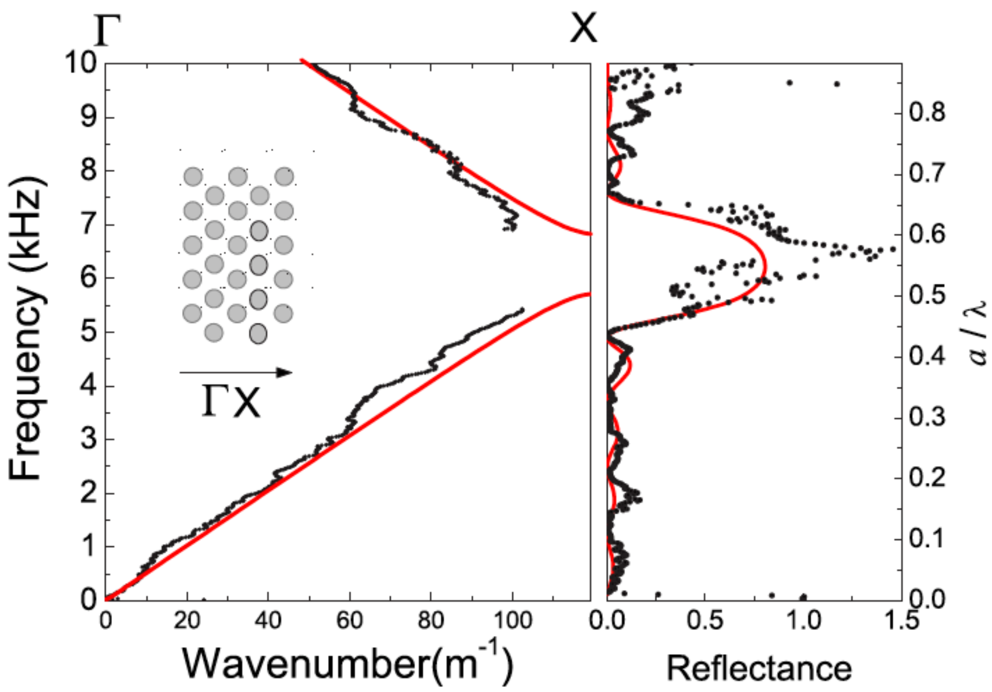
Homogenization



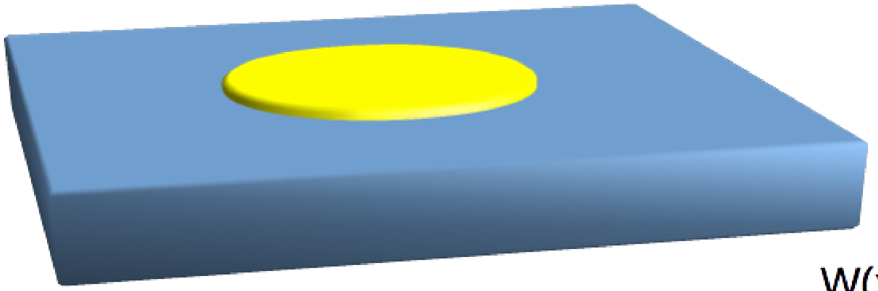


Homogenization





Flexural Waves in Thin Elastic Plates



W(x,y)

$$(D_b\nabla^4 - \rho_b h_b \omega^2) W(x,y) = 0$$

$$k^4 = \frac{\rho h}{D}\omega^2$$

Multiple Scattering of Flexural Waves

For a scattering problem, the incident field is expressed as

$$W_0 = \sum_q \left[A_q^J J_q(k_b r) + A_q^I I_q(k_b r)\right] e^{iq\theta}, \qquad (16)$$

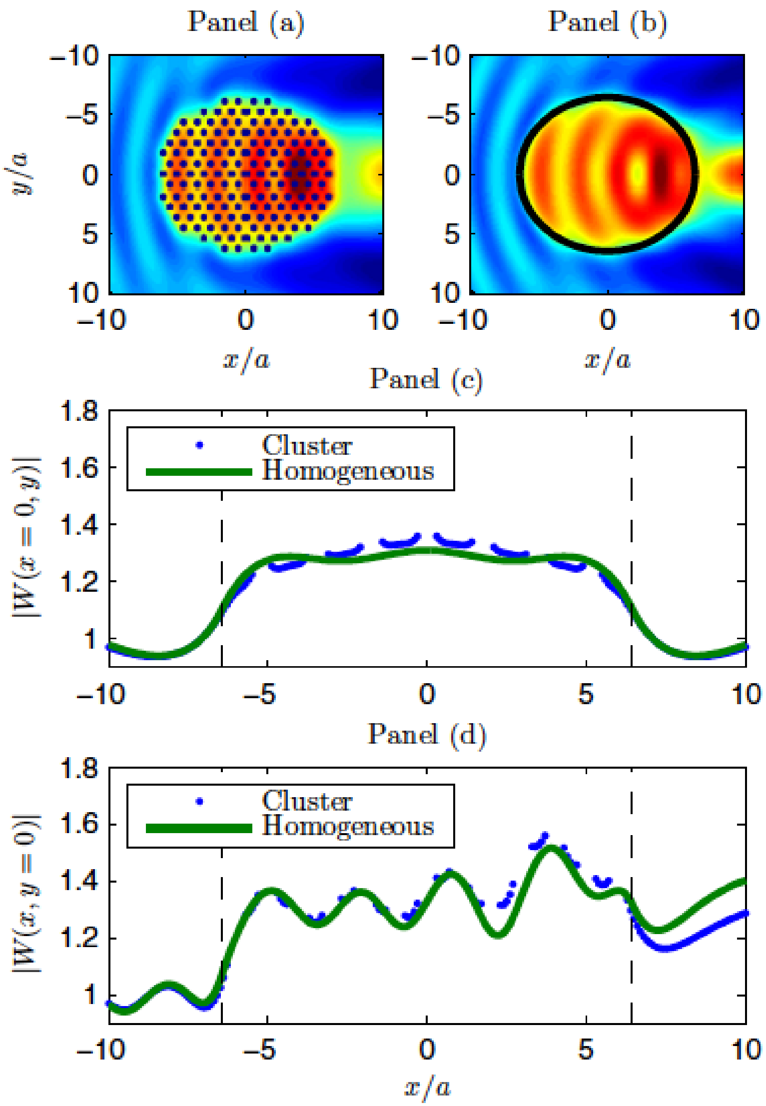
while the scattered field is given by

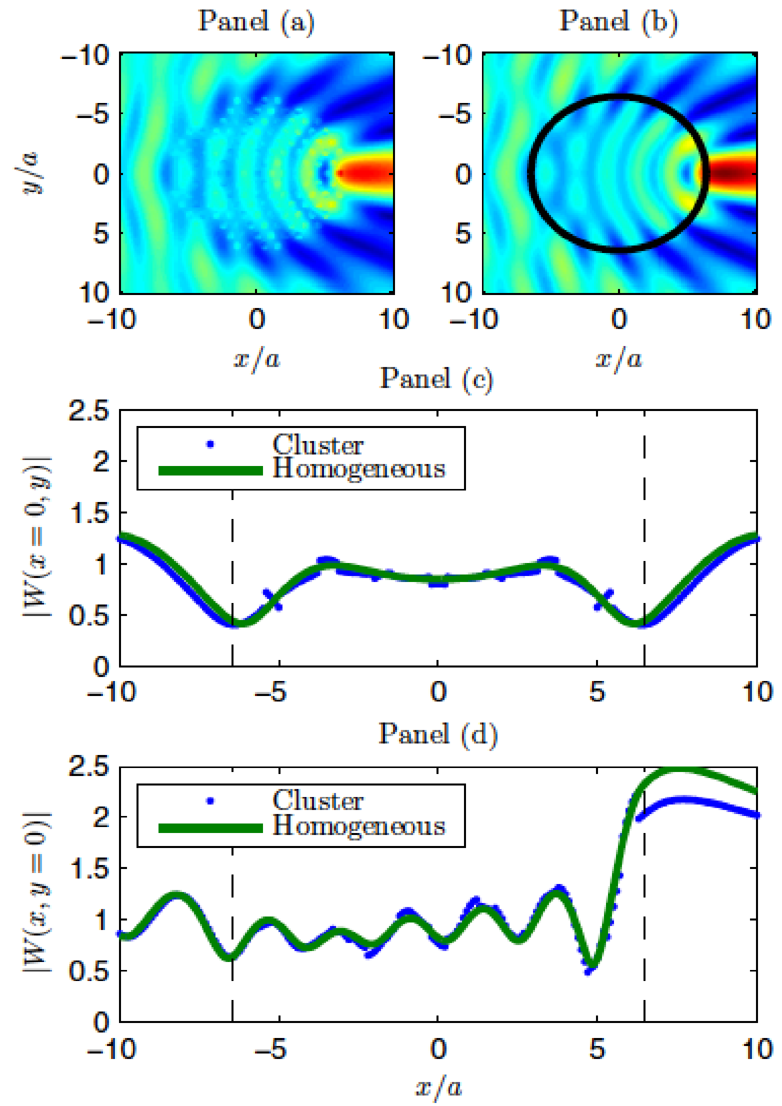
$$W_{sc} = \sum_q \left[B_q^H H_q(k_b r) + B_q^K K_q(k_b r)\right] e^{iq\theta}. \qquad (17)$$

If the scatterer is a circular inhomogeneity of radius R_a we have that, inside the scatterer ($r < R_a$), since there are no sources, the field is expressed as

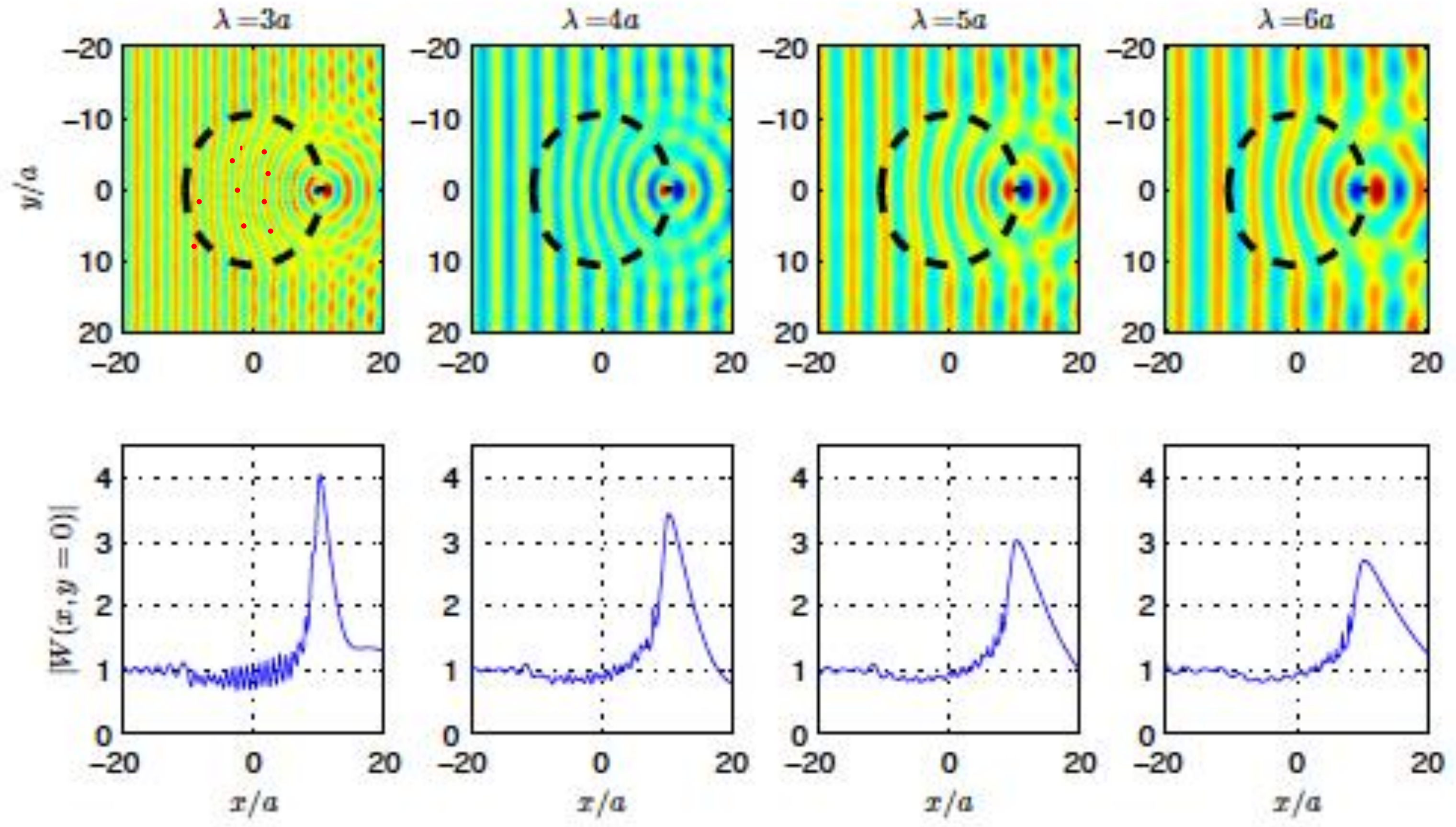
$$W_i = \sum_q \left[C_q^J J_q(k_a r) + C_q^I I_q(k_a r)\right] e^{iq\theta}. \qquad (18)$$

Multiple Scattering of Flexural Waves



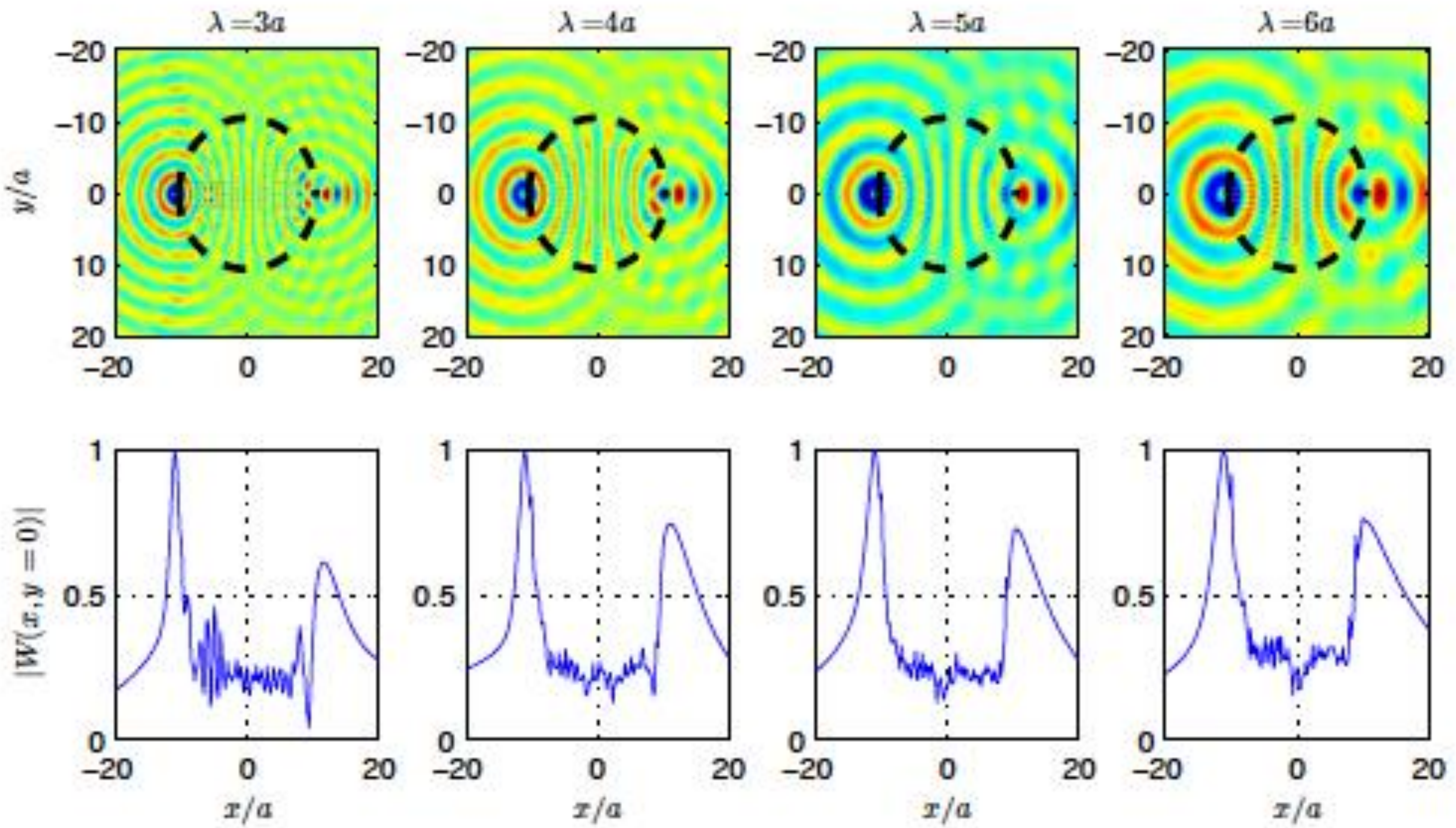


Luneburg Lens for Flexural Waves

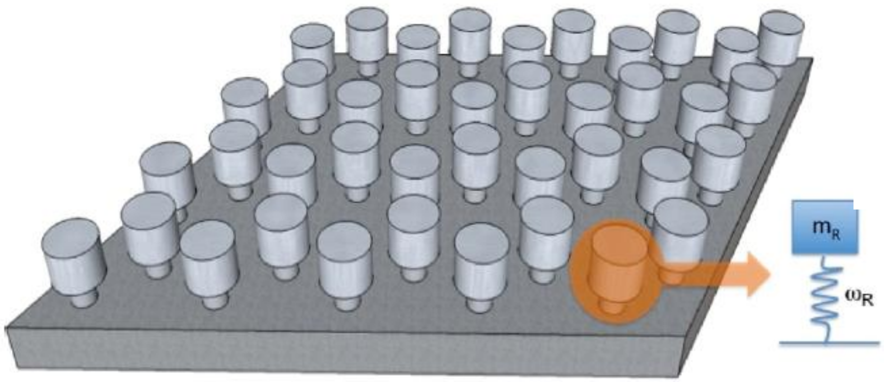


Multiple Scattering of Flexural Waves

Maxwell Lens for Flexural Waves



Multiple Scattering of Point Scatterers



$$\left(\nabla^4 - \omega^2 \frac{\rho h}{D}\right) W_1(\boldsymbol{r}) = \sum_{\boldsymbol{R}_n} \sum_{\alpha} t_\alpha W_1(\boldsymbol{R}_{n\alpha}) \delta(r - \boldsymbol{R}_{n\alpha})$$

The problem of multiple scattering is solved by setting up a system of self-consistent equations, so that the solution for the field $W_1(\boldsymbol{r})$ under some incident excitation $\psi_0(\boldsymbol{r})$ is given by[35]

$$W_1(\boldsymbol{r}) = \psi_0(\boldsymbol{r}) + \sum_{\alpha} T_\alpha \psi_e(\boldsymbol{R}_\alpha) G_0(\boldsymbol{r} - \boldsymbol{R}_\alpha), \qquad (44)$$

where $\psi_e(\boldsymbol{R}_\alpha)$ is called the "external" field and is the incident field on the scatterer α; thus

$$\psi_e(\boldsymbol{R}_\alpha) = \psi_0(\boldsymbol{R}_\alpha) + \sum_{\beta \neq \alpha} T_\beta \psi_e(\boldsymbol{R}_\beta) G_0(\boldsymbol{R}_\alpha - \boldsymbol{R}_\beta). \qquad (45)$$

$$(\nabla^4 - k_0^4)W(\boldsymbol{r}) = \sum_{\alpha=1} t_\alpha \delta(\boldsymbol{r} - \boldsymbol{R}_\alpha)W(\boldsymbol{R}_\alpha)$$

$$W(\boldsymbol{r}) = \psi_0(\boldsymbol{r}) + \sum_{\alpha=1}^{N} B_\alpha G(\boldsymbol{r} - \boldsymbol{R}_\alpha)$$

$$(\nabla^4 - k_0^4)\psi_0(\boldsymbol{r}) = 0$$

$$(\nabla^4 - k_0^4)G(\boldsymbol{r} - \boldsymbol{R}_\alpha) = \delta(\boldsymbol{r} - \boldsymbol{R}_\alpha)$$

$$\sum_{\alpha=1}^{N} B_\alpha \delta(\boldsymbol{r} - \boldsymbol{R}_\alpha) = \sum_{\alpha=1}^{N} \delta(\boldsymbol{r} - \boldsymbol{R}_\alpha) t_\alpha \left(\psi_0(R_\alpha) + \sum_{\beta=1}^{N} B_\beta G(\boldsymbol{R}_\alpha - \boldsymbol{R}_\beta) \right)$$

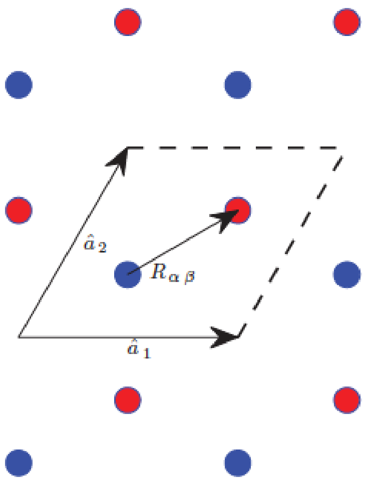
$$\sum_{\beta=1}^{N} M_{\alpha\beta} B_\beta = \psi_0(\boldsymbol{R}_\alpha), \ \alpha = 1,2, \dots, N$$

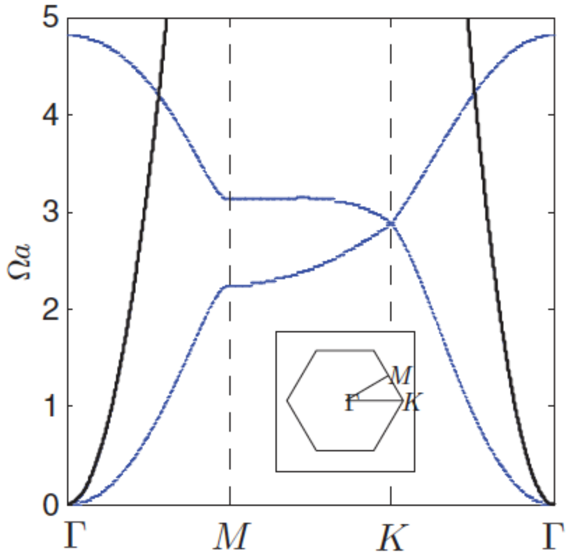
$$M_{\alpha\beta} = \delta_{\alpha\beta} t_\alpha^{-1} - G(\boldsymbol{R}_\alpha - \boldsymbol{R}_\beta)$$

Diagonal term: influence of each scatterer to the cluster

Non-diagonal term: interaction between scatterers

Multiple Scattering of Point Scatterers



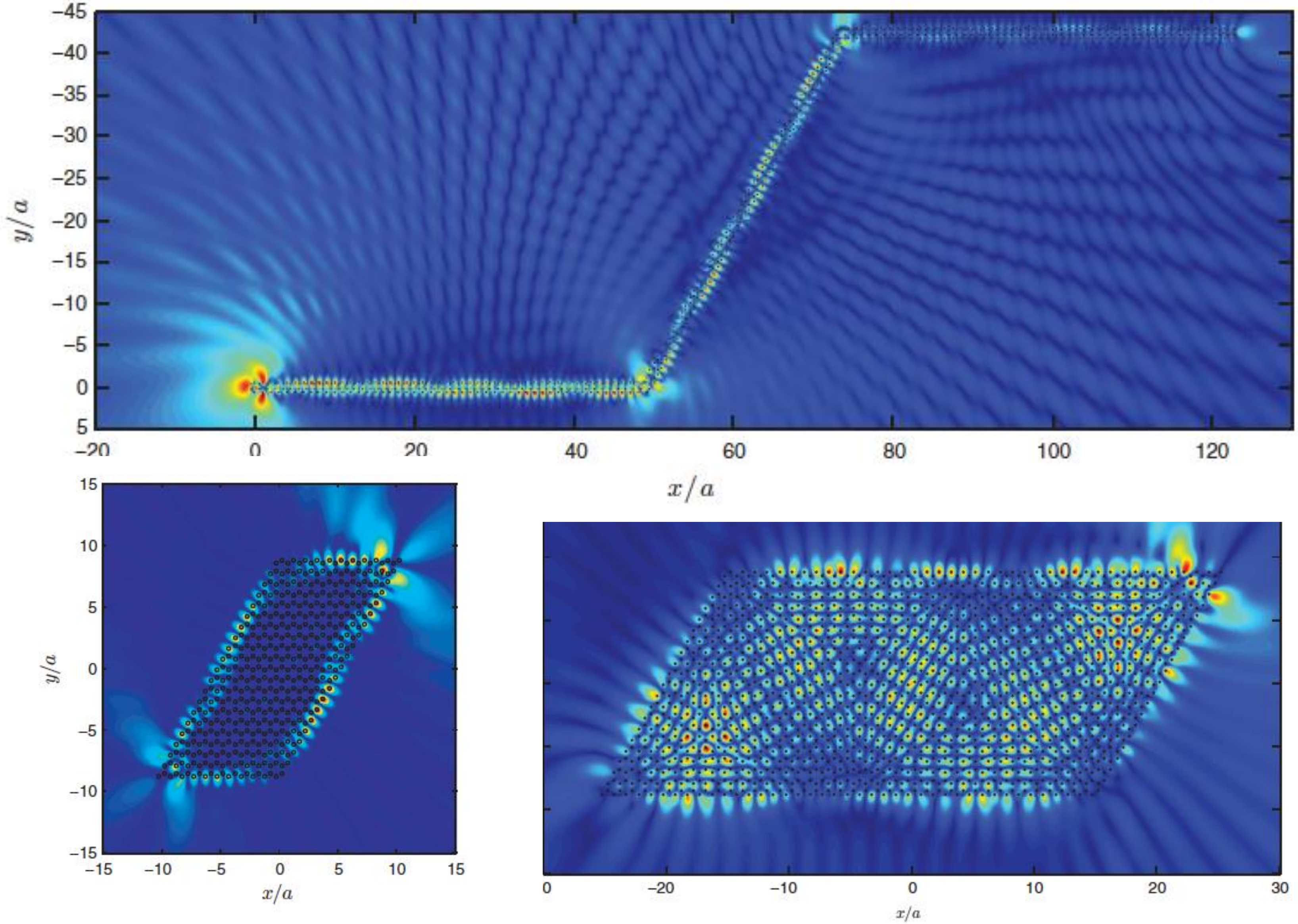


$$\begin{vmatrix} 1-\chi_{\alpha\alpha} & -\chi_{\alpha\beta} \\ -\chi^*_{\alpha\beta} & 1-\chi_{\alpha\alpha} \end{vmatrix} = 0.$$

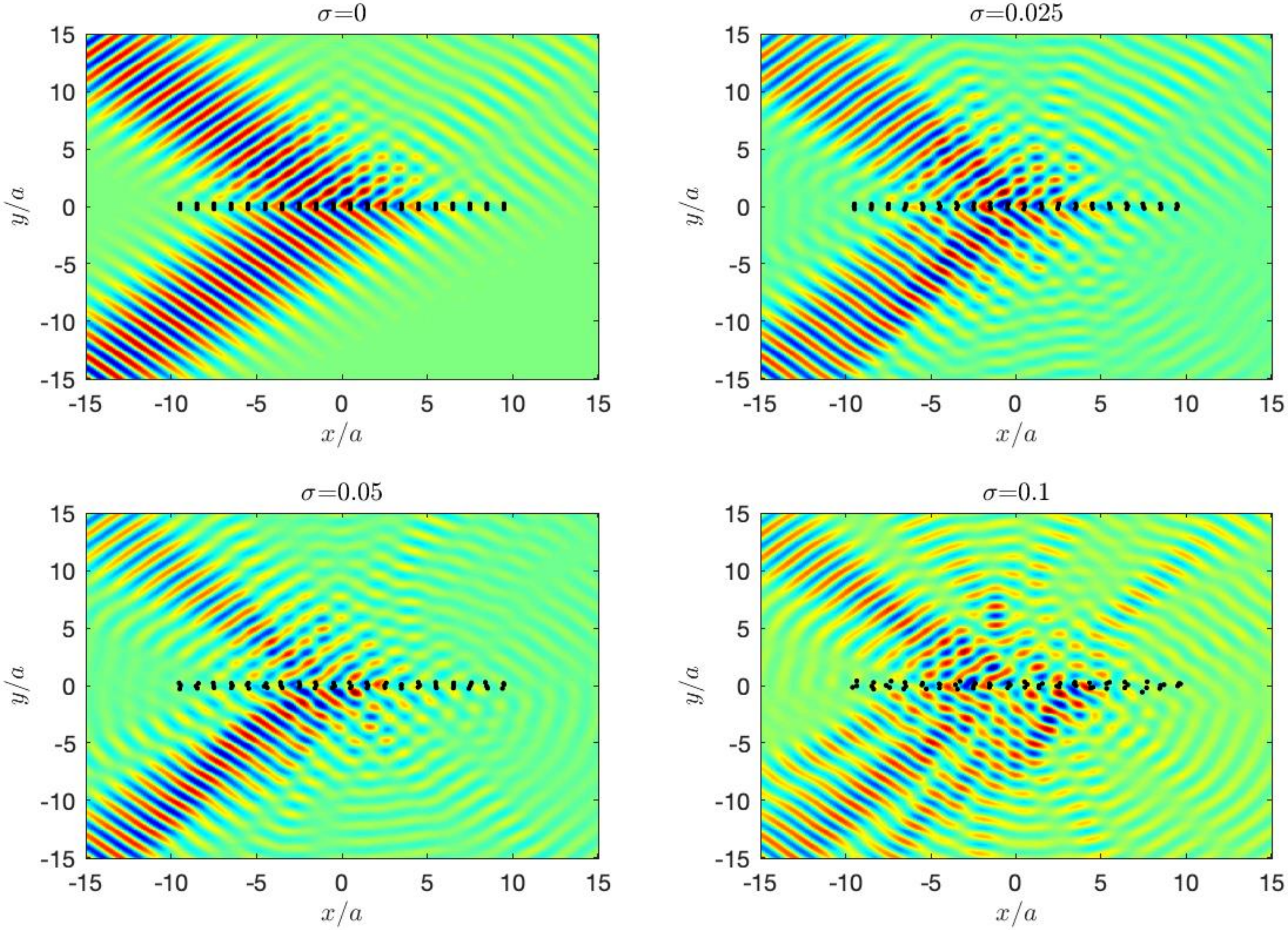
$$\begin{vmatrix} -\delta\Omega a/c_D & \delta \boldsymbol{K}\cdot(\hat{\boldsymbol{x}}+i\,\hat{\boldsymbol{y}}) \\ \delta \boldsymbol{K}\cdot(\hat{\boldsymbol{x}}-i\,\hat{\boldsymbol{y}}) & -\delta\Omega a/c_D \end{vmatrix} = 0$$

$$\frac{1}{c_D} = \frac{1}{|S_{\alpha\beta}|\gamma_R\Omega_D^3 a^3} + \Omega_D a \frac{S_{\alpha\alpha}}{|S_{\alpha\beta}|}$$

Multiple Scattering of Point Scatterers



Multiple Scattering of Point Scatterers



Multiple Scattering of Point Scatterers

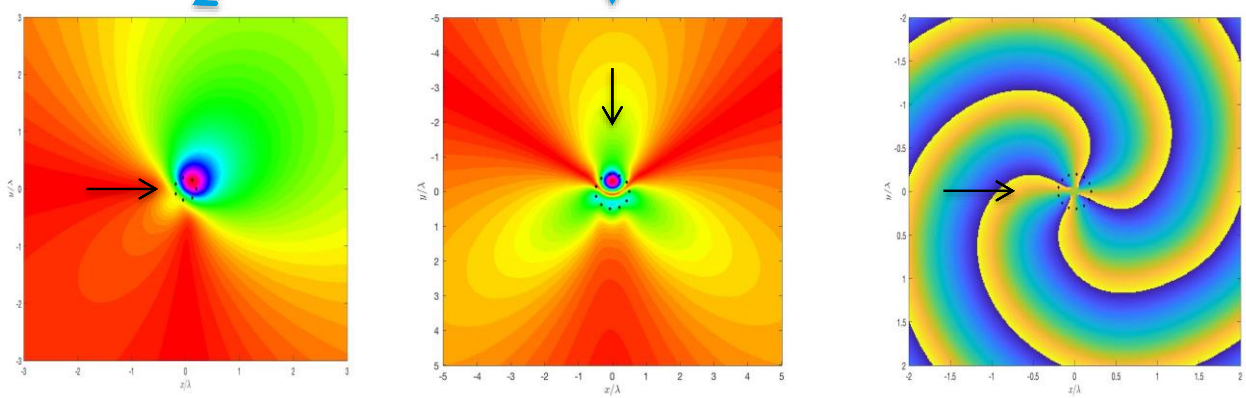
$$\Psi = A_0 \Psi_{in} + \sum_{b=1}^{N} B_b G(r - R_b) = \sum_{q=-\infty}^{\infty} A_q f_q(r) e^{iqq}$$

$$A_q = \sum_{b=1}^{N} S_{qb} B_b$$

$$\Psi \approx \cos(3\pi/4) - \cos(\vartheta - \pi/4)$$

$$\Psi \approx \cos(\frac{3}{2}\vartheta)$$

$$\Psi \approx e^{iQ\vartheta}$$



MODE RESONANCES

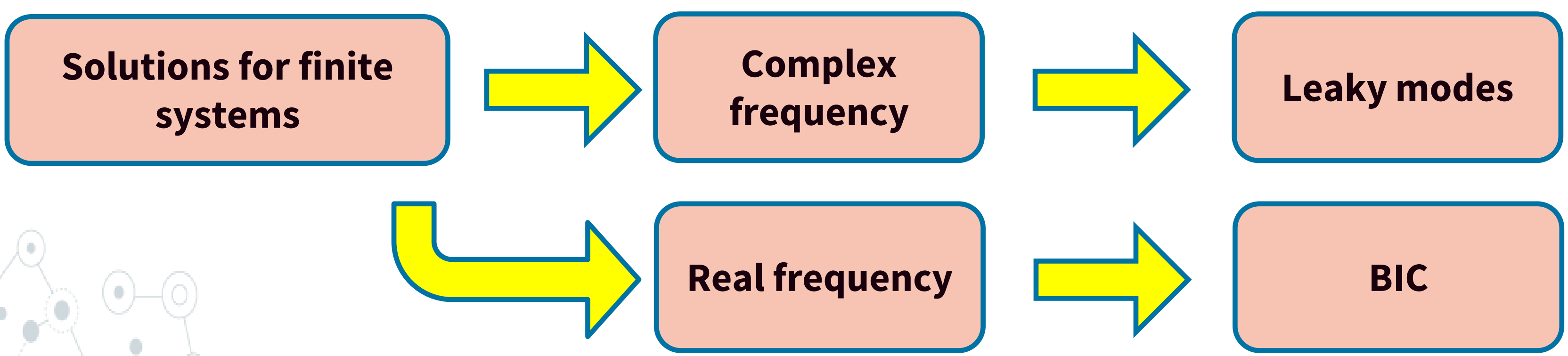
Resonances of a cluster: presence of scattered field without need of having incident field

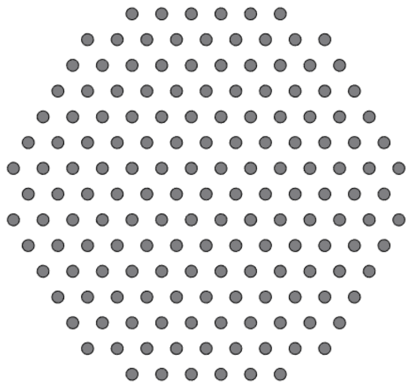
$$\sum_{\beta=1}^{N} M_{\alpha\beta} B_\beta = \psi_0(\boldsymbol{R}_\alpha),\ \ \alpha = 1,2,\ldots,N$$

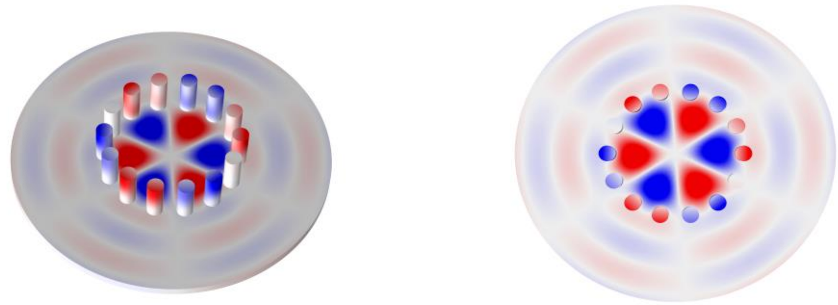
$$\psi_0(\boldsymbol{r}) = 0$$

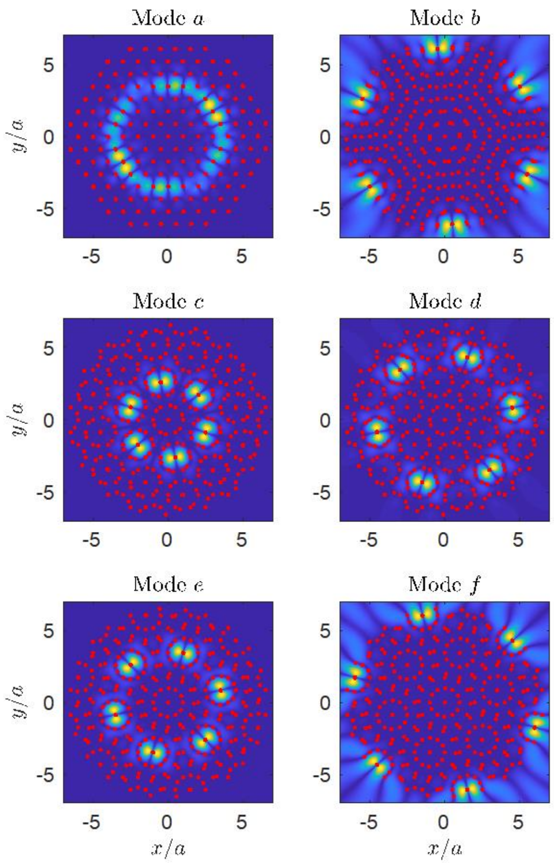
Eigenmodes of the cluster

$$\sum_{\beta=1}^{N} M_{\alpha\beta} B_\beta = 0 \quad \Rightarrow \quad |\boldsymbol{M}_{\alpha\beta}| = 0 \ \rightarrow \exists\lambda = 0$$

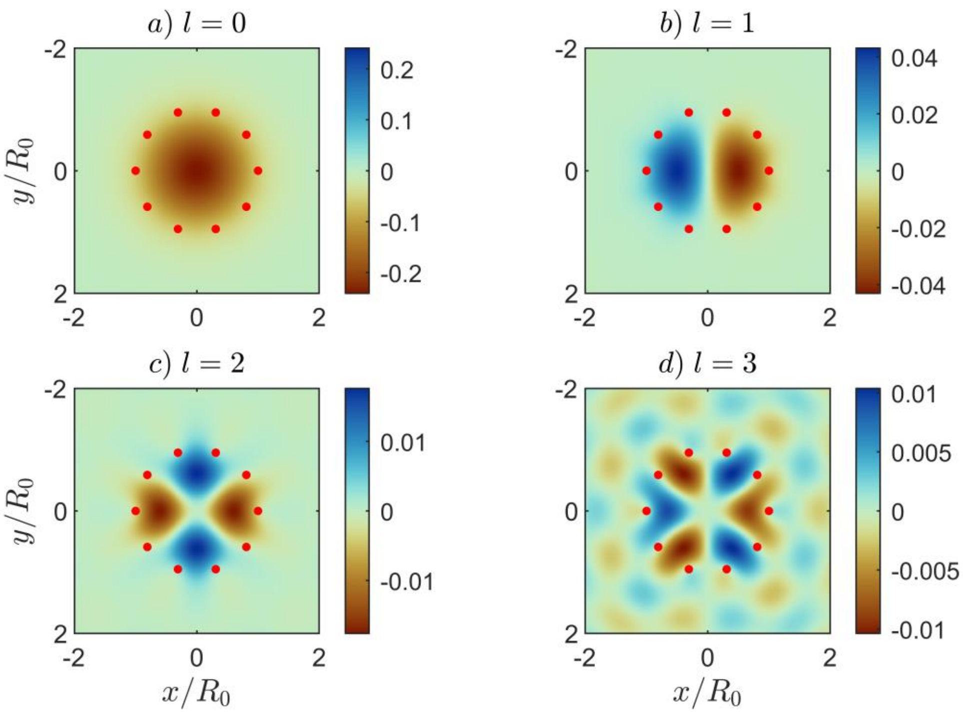








Mode a
Mode b
Mode c
Mode d
Mode e
Mode f
y/a
x/a
5
0
-5



$a)\ l=0$
$b)\ l=1$
$c)\ l=2$
$d)\ l=3$
y/R_0
x/R_0
-2
0
2
0.2
0.1
-0.1
-0.2
0.04
0.02
-0.02
-0.04
0.01
-0.01
0.005
-0.005

Thank you!!!