Metamaterials: time-domain numerical methods for dispersive and nonlinear media

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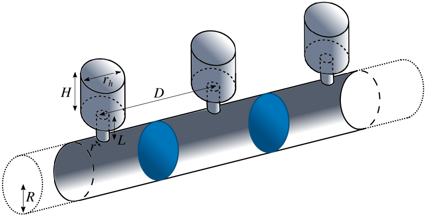
Part I

Introduction

Example 1/3: resonant waveguide



- guide connected with membranes and Helmholtz resonators (Fang et al, 2006)

  - $1D$ plane mode, $\lambda \gg D$: homogenization

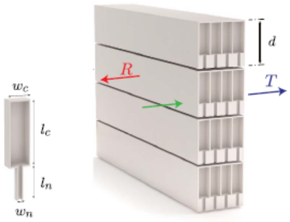
$$\partial_x \left(\frac{1}{\hat{\rho}(\omega)} \partial_x \hat{p} \right) + \omega^2 \, \hat{\kappa}^{-1}(\omega) \, \hat{p} = 0$$

$$\left| \begin{array}{l} \hat{\rho}(\omega) = \rho_0 \left(1 - \dfrac{\Omega_\rho^2}{\omega^2 - \omega_\rho^2} \right), \qquad \hat{\kappa}^{-1}(\omega) = \kappa_0^{-1} \left(1 - \dfrac{\Omega_\kappa^2}{\omega^2 - \omega_\kappa^2 - i\gamma\omega} \right) \\ \omega_\kappa = c_0 \sqrt{\dfrac{A}{LV}}, \quad \Omega_\kappa = \omega_\kappa \sqrt{\dfrac{V}{SD}}, \quad \Omega_\rho = \sqrt{\dfrac{K}{\rho_m}} \end{array} \right.$$

✓ frequency-dependence of the effective density and compressibility

✓ large amplitude: nonlinear Burgers' equation (Sugimoto, 1995)

Example 2/3: resonant metasurface



📚 Romero-Garcia et al, CR Physique (2020)

- reduction of size: thin row of resonant scatterers

  - subwavelength resonators ($d \ll \lambda$)

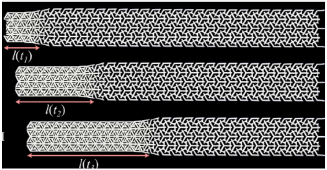
  - matched-asymptotic expansions + homogenisation = interface (simpler)

$$\begin{cases} [\![\hat{p}]\!]_a = \boldsymbol{B}.\langle \nabla \hat{p} \rangle_a \\ [\![\hat{\boldsymbol{v}}.\boldsymbol{n}]\!]_a = \boldsymbol{C} : \langle \nabla \hat{\boldsymbol{v}} \rangle_a + \mathscr{D}_\infty(\omega) \langle \text{div } \hat{\boldsymbol{v}} \rangle_a \end{cases}$$

$$\mathscr{D}_\infty(\omega) = \alpha_0 - \sum_{r \geq 1} {\alpha_r}^2 \frac{\omega^2}{\omega^2 - {\omega_r}^2}$$

✓ frequency-dependent jump conditions

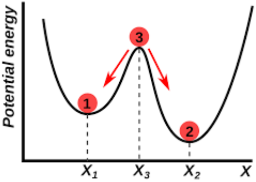
Example 3/3: deployable structures



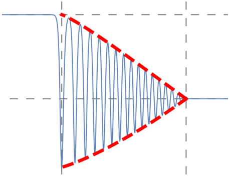
Khajehtourian-Kochmann EML (2020)

- continuum approximation
  - ✓ nonlinearities: finite deformations + bistability
  - ✓ dispersion: travelling wave

bistability



dispersion



Position of the problem

- dispersion, frequency-dependant parameters:

  - natural formulation in the frequency domain

  - in time domain: convolution products

$$\hat{\rho}(\omega) \times \hat{v}(\boldsymbol{x},\omega) \mapsto \rho(t) \star v(\boldsymbol{x},t)$$

  ✗ non-local in time, huge storage

- nonlinearities:

  ✗ generation of harmonics = difficulties in the frequency domain

  ✗ non-smooth solutions

  ✗ not uniqueness of solution

- interface conditions:

  ✗ available numerical tools?

  ✗ accuracy + efficiency?

Sketch of this Lecture

Objective: toolbox for time-domain simulations in metamaterials

- time-domain formulations:

  - dispersion, nonlinearity

- numerical methods:

  - schemes, splitting, interfaces

- illustration through many examples

remarks:

✓ no justification of models (topic of other Lectures!)

✓ very incomplete panel of numerical methods

✓ algorithms in 1D

Part II

Mathematical modeling

How to formulate resonant parameters in time?

- frequency-domain acoustics

$$\begin{cases} i\omega\,\hat{\rho}(\omega)\,\hat{\boldsymbol{v}} + \nabla\hat{p} = \mathbf{0} \\ i\omega\,\hat{\kappa}^{-1}(\omega)\,\hat{p} + \operatorname{div}\,\hat{\boldsymbol{v}} = 0 \end{cases}$$

frequency-dependent parameters

$$\hat{\rho}(\omega) = \rho_0\left(1 - \frac{\Omega_\rho^2}{\omega^2 - \omega_\rho^2}\right), \qquad \hat{\kappa}^{-1}(\omega) = \kappa_a^{-1}\left(1 - \frac{\Omega_\kappa^2}{\omega^2 - \omega_\kappa^2 - i\gamma\omega}\right)$$

- strategy:

formalism of auxiliary fields = additional fields $\rightsquigarrow$ local-in-time PDE

✓ Maxwell equations (Gralak-Tip 10, Vinoles 16, Cassier-Joly 17)

✓ fractional derivatives (Matignon 10)

Auxiliary fields

- conservation of momentum + effective density

$$\begin{cases} i\omega\,\hat{\rho}\,\hat{\boldsymbol{v}} + \nabla\hat{p} = \mathbf{0} \\ \hat{\rho}(\omega) = \rho_a\left(1 - \dfrac{\Omega_\rho^2}{\omega^2 - \omega_\rho^2}\right) \end{cases}$$

auxiliary field $\hat{\boldsymbol{w}}$

$$\begin{cases} i\omega\,\rho_a\,\hat{\boldsymbol{v}} + i\omega\,\rho_a\,\Omega_\rho^2\,\hat{\boldsymbol{w}} + \nabla\hat{p} = \mathbf{0} \\ (-\omega^2 + \omega_\rho^2)\,\hat{\boldsymbol{w}} = \hat{\boldsymbol{v}} \end{cases}$$

inverse Fourier transform in time

$$\begin{cases} \rho_a\dfrac{\partial\boldsymbol{v}}{\partial t} + \rho_a\,\Omega_\rho^2\dfrac{\partial\boldsymbol{w}}{\partial t} + \nabla p = \mathbf{0} \\ \dfrac{\partial^2\boldsymbol{w}}{\partial t^2} + \omega_\rho^2\,\boldsymbol{w} = \boldsymbol{v} \end{cases}$$

- conservation of mass + effective compressibility: auxiliary field r

$$\begin{cases} i\omega\,\hat{\kappa}^{-1}(\omega)\,\hat{p} + \text{div }\hat{\boldsymbol{v}} = 0 \\ \hat{\kappa}^{-1}(\omega) = \kappa_a^{-1}\left(1 - \dfrac{\Omega_\kappa^2}{\omega^2 - \omega_\kappa^2 - i\,\gamma\,\omega}\right) \end{cases} \Rightarrow \begin{cases} \dfrac{\partial p}{\partial t} + \Omega_\kappa^2\dfrac{\partial r}{\partial t} + \kappa_a\text{div }\boldsymbol{v} = 0 \\ \dfrac{\partial^2 r}{\partial t^2} + \gamma\dfrac{\partial r}{\partial t} + \omega_\kappa^2\,r = p \end{cases}$$

First-order augmented system

- additional auxiliary fields $\boldsymbol{u} = \frac{\partial \boldsymbol{w}}{\partial t}$ and $q = \frac{\partial r}{\partial t}$

$$\begin{cases} \rho(t) \star \dfrac{\partial \boldsymbol{v}}{\partial t} + \nabla p = \boldsymbol{0} \\ \hat{\kappa}^{-1}(t) \star \dfrac{\partial p}{\partial t} + \operatorname{div} \boldsymbol{v} = 0 \end{cases} \rightarrow \begin{cases} \dfrac{\partial \boldsymbol{v}}{\partial t} + \dfrac{1}{\rho_a} \nabla p + \Omega_\rho^2 \, \boldsymbol{u} = \boldsymbol{0} \\ \dfrac{\partial p}{\partial t} + \kappa_a \operatorname{div} \boldsymbol{v} + \Omega_\kappa^2 \, q = 0 \\ \dfrac{\partial \boldsymbol{u}}{\partial t} - \boldsymbol{v} + \omega_\rho^2 \, \boldsymbol{w} = \boldsymbol{0} \\ \dfrac{\partial \boldsymbol{w}}{\partial t} - \boldsymbol{u} = \boldsymbol{0} \\ \dfrac{\partial q}{\partial t} - p + \gamma \, q + \omega_\kappa^2 \, r = 0 \\ \dfrac{\partial r}{\partial t} - q = 0 \end{cases}$$

- compact writing $\boldsymbol{U} = \left(\boldsymbol{v}, \, p, \, \boldsymbol{u}, \, \boldsymbol{w}, \, q, \, r\right)^\top$

$$\boxed{\frac{\partial}{\partial t} \boldsymbol{U} + \sum_{j=1}^{d} \boldsymbol{A}_j \, \frac{\partial}{\partial x_j} \boldsymbol{U} = \boldsymbol{S} \, \boldsymbol{U} \qquad (\boldsymbol{x} \in \mathbb{R}^d, \, t > 0)}$$

Formulation of conservation laws

- first-order system with source term

$$\frac{\partial}{\partial t}\boldsymbol{U} + \sum_{j=1}^{d} \frac{\partial}{\partial x_j}\boldsymbol{f}_j(\boldsymbol{U}) = \boldsymbol{g}(\boldsymbol{U}), \qquad (\boldsymbol{x} \in \mathbb{R}^d,\ t > 0)$$

$\boldsymbol{f}_j$ flux function, $\boldsymbol{g}$ relaxation + diffusion term

divergence theorem: variation of $\boldsymbol{U}$ = flux on boundary + variation in volume

- 1D examples:

  - linear advection: $\partial_t u + a\partial_x u = 0$
  - linear advection-reaction: $\partial_t u + a\partial_x u = -\alpha\, u$
  - inviscid Burgers: $\partial_t u + \partial_x \left(\frac{u^2}{2}\right) = 0$
  - nonlinear elasticity: $\partial_t \boldsymbol{U} + \partial_x \boldsymbol{f}(\boldsymbol{U}) = \mathbf{0}$

$$\boldsymbol{U} = (\varepsilon,\ v)^\top, \quad \boldsymbol{f}(\boldsymbol{U}) = (-v, -\frac{1}{\rho}\sigma(\varepsilon))^\top, \qquad \text{with } \sigma(\varepsilon) \text{ constitutive law}$$

Hyperbolic PDEs

$$\frac{\partial}{\partial t}\boldsymbol{U} + \sum_{j=1}^{d} \frac{\partial}{\partial x_j}\boldsymbol{f}_j(\boldsymbol{U}) = \boldsymbol{g}(\boldsymbol{U})$$

- jacobians $\boldsymbol{A}_j = \frac{\partial}{\partial \boldsymbol{U}}\boldsymbol{f}_j(\boldsymbol{U})$

- space dimension $d > 1$: $\boldsymbol{k} = (k_1, \cdots, k_d)^\top \in \mathbb{R}^d$, $\boldsymbol{A}(\boldsymbol{k}) = \sum_{j=1}^{d} k_j \boldsymbol{A}_j$

Property (hyperbolicity)

The system is hyperbolic if the matrix $\boldsymbol{A}(\boldsymbol{k})$ is diagonalizable for all $\boldsymbol{k} \in \mathbb{R}^d \setminus \{\boldsymbol{0}\}$ with real eigenvalues.

✓ well-posed problem + finite velocity of energy

✓ example: 1D acoustics

$$\left| \begin{array}{l} \frac{\partial}{\partial t}\boldsymbol{U} + \boldsymbol{A}\frac{\partial}{\partial x}\boldsymbol{U} = \boldsymbol{0} \\ \boldsymbol{U} = \begin{pmatrix} v \\ p \end{pmatrix}, \qquad \boldsymbol{A} = \begin{pmatrix} 0 & \frac{1}{\rho} \\ \kappa & 0 \end{pmatrix} \end{array} \right.$$

$\mathrm{Sp}(\boldsymbol{A}) = \left\{\pm\sqrt{\frac{\kappa}{\rho}}\right\} \quad \Rightarrow$ hyperbolic

Property for nonlinear waves

1D homogeneous nonlinear system

$$\begin{cases} \dfrac{\partial}{\partial t}\boldsymbol{U} + \dfrac{\partial}{\partial x}\boldsymbol{f}(\boldsymbol{U}) = \boldsymbol{0} \\ \boldsymbol{U}(x,0) = \boldsymbol{U}_0(x) \end{cases}$$

Property (shocks)

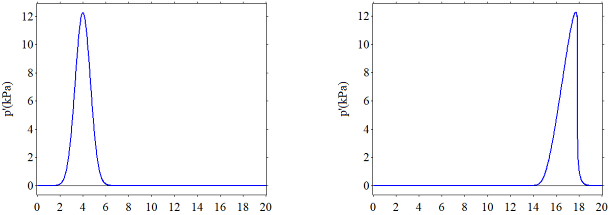
The following two properties hold:

(i) *discontinuities (ie shocks) develop in finite time* $t^\star$

(ii) *the shock speed* s *satisfies the Rankine-Hugonot conditions* $[\![\boldsymbol{f}(\boldsymbol{U})]\!] = s\,[\![\boldsymbol{U}]\!]$

✓ example: Burgers 1D

$$\partial_t u + \partial_x\left(\frac{u^2}{2}\right) = 0, \qquad t^\star = -\frac{1}{\min u_0'(x)}$$



Part III

Numerical modeling

Conservative schemes

1D homogeneous nonlinear system

$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{f}(\boldsymbol{U}) = \boldsymbol{0}$$

- finite-volume schemes:

  - uniform grid in space and time $x_i = i\,\Delta x$, $t_{n+1} = t_n + \Delta t$

  - integration on $[x_{i_1/2}, x_{i+1/2}] \times [t_n, t_{n+1}]$

  - approximations: mean value $\boldsymbol{U}_i^n$, numerical flux $\boldsymbol{F}^n_{i+1/2}$

$$\boldsymbol{U}_i^n \approx \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \boldsymbol{U}(x, t_n)\,dx \approx \boldsymbol{U}(x_i, t_n), \quad \boldsymbol{F}^n_{i+1/2} \approx \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} \boldsymbol{f}(\boldsymbol{U}(x_{i+1/2}, t))\,dt$$

  - discrete conservative form

$$\boxed{\boldsymbol{U}_i^{n+1} = \boldsymbol{U}_i^n - \frac{\Delta t}{\Delta x}\left(\boldsymbol{F}^n_{i+1/2} - \boldsymbol{F}^n_{i-1/2}\right)}$$

✓ estimation of $\boldsymbol{F}^n_{i+1/2} \mapsto$ families of schemes

Some examples (1D linear acoustics)

- notations:

  - conservation law

$$\frac{\partial}{\partial t} \boldsymbol{U} + \boldsymbol{A} \frac{\partial}{\partial x} \boldsymbol{U} = \boldsymbol{0}$$

  - numerical flux

$$\boldsymbol{F}^n_{i+1/2} \approx \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} \boldsymbol{A}\, \boldsymbol{U}(x_{i+1/2}, t)\, dt$$

- elementary schemes:

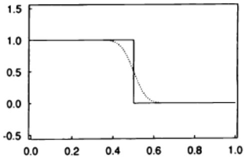
Godunov scheme ($|\boldsymbol{A}| = \boldsymbol{R}\, |\Lambda|\, \boldsymbol{R}^{-1}$)

$$\boldsymbol{F}^n_{i+1/2} = \frac{1}{2} \left(\boldsymbol{A}\, \boldsymbol{U}^n_i + \boldsymbol{A}\, \boldsymbol{U}^n_{i+1} \right) - \frac{1}{2} |\boldsymbol{A}| \left(\boldsymbol{U}^n_{i+1} - \boldsymbol{U}^n_i \right)$$

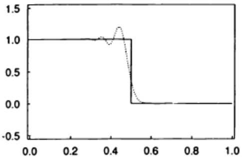
Lax-Wendroff scheme

$$\boldsymbol{F}^n_{i+1/2} = \frac{1}{2} \left(\boldsymbol{A}\, \boldsymbol{U}^n_i + \boldsymbol{A}\, \boldsymbol{U}^n_{i+1} \right) - \frac{\Delta t}{2\, \Delta x} \boldsymbol{A}^2 \left(\boldsymbol{U}^n_{i+1} - \boldsymbol{U}^n_i \right)$$

Godunov



Lax-Wendroff



Numerical analysis

- local truncation error

  - error due to one step of discretization

  - example: discretization of the ODE $u' - \lambda u = 0$

$$
\begin{array}{llcl}
\text{Euler explicit} & 0 & = & \dfrac{u^{n+1} - u^n}{\Delta t} - \lambda u^n \\
\text{exact solution} & L(t_n) & = & \dfrac{u(t_{n+1}) - u(t_n)}{\Delta t} - \lambda u(t_n) \\
\text{Taylor expansion} & & = & \dfrac{u(t_n) + \Delta t\, u'(t_n) + O(\Delta t^2) - u(t_n)}{\Delta t} - \lambda u(t_n) \\
\text{ODE} & & = & \underbrace{u'(t_n) - \lambda u(t_n)}_{=0} + O(\Delta t)
\end{array}
$$

- stability

  - linear amplification of the errors due to discretization

  - Courant-Friedrichs-Lewy condition (CFL):

$$\max c \frac{\Delta t}{\Delta x} \leq 1$$

Theorem (Lax-Richtmyer)

Local error of order q + stability $\Rightarrow$ global error of order q

Importance of the conservative form

- conservative form

$$\partial_t u + \partial_x \left(\frac{u^2}{2} \right) = 0$$

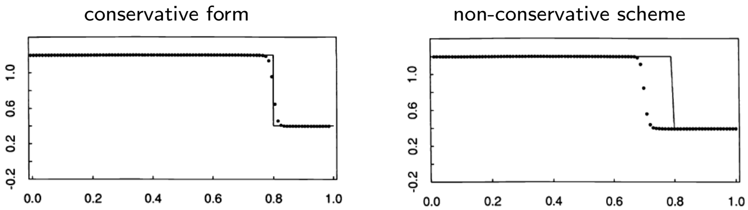
  - conservative Godunov scheme

- non-conservative form (smooth solution):

$$\partial_t u + u \, \partial_x u = 0$$

  - non-conservative scheme

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x} u_i^n \left(u_i^n - u_{i-1}^n \right)$$



(LeVeque, 1993)

✓ conservative form → Rankine-Hugoniot → correct shock speed

To go further

- linear PDE: high-order ($k \geq 4$)

  Arbitrary DERivatives = ADER schemes

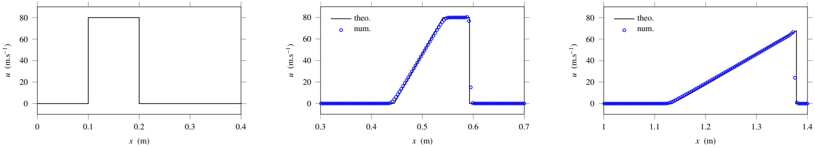
  generalization of Lax-Wendroff

  explicit, two-step, spatially centered

- nonlinear PDE

  ✗ non-smoothness + non-uniqueness

  ✓ high-order (in the smooth part) and low-order (near the shock)



Berjamin et al, Acustica (2017)

Relaxation and dissipation

- nonlinear hyperbolic system with linear source term

$$\boxed{\frac{\partial}{\partial t}\boldsymbol{U} + \frac{\partial}{\partial x}\boldsymbol{f}(\boldsymbol{U}) = \boldsymbol{S}\,\boldsymbol{U}}$$

✗ naive direct discretization: $\Delta t \leq \min\left(\frac{\Delta x}{c_{\max}}, \frac{2}{\varrho(\boldsymbol{S})}\right)$

Strang splitting (propagation and relaxation)

$$\left|\begin{array}{ll} \frac{\partial}{\partial t}\boldsymbol{U} + \frac{\partial}{\partial x}\boldsymbol{f}(\boldsymbol{U}) = \boldsymbol{0} & (\boldsymbol{H}_p) \\ \frac{\partial}{\partial t}\boldsymbol{U} = \boldsymbol{S}\,\boldsymbol{U} & (\boldsymbol{H}_r) \end{array}\right.$$

$$\boxed{\boldsymbol{U}^{n+1} = \boldsymbol{H}_r\left(\frac{\Delta t}{2}\right) \circ \boldsymbol{H}_p(\Delta t) \circ \boldsymbol{H}_r\left(\frac{\Delta t}{2}\right)\,\boldsymbol{U}^n}$$

- propagation step: finite-volume scheme with flux limiters

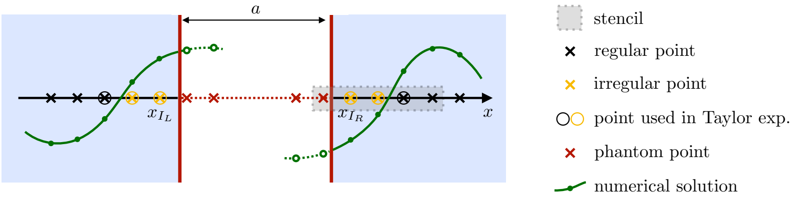
$$\boldsymbol{U}_i^{n+1} = \boldsymbol{U}_i^n - \frac{\Delta t}{\Delta x}\left(\boldsymbol{F}_{i+1/2}^n - \boldsymbol{F}_{i-1/2}^n\right), \qquad \Delta t \leq \frac{\Delta x}{c_{\max}}$$

- relaxation step: exact solution

$$\boldsymbol{H}_r(\tau)\,\boldsymbol{U}_i = \exp(\boldsymbol{S}\tau)\,\boldsymbol{U}_i$$

Interfaces

- difficulties:
  - ✗ geometry: structured grid (poor geometry) vs unstructured grid (CFL $\Delta t \searrow$)
  - ✗ physics: nonlinear contacts, frequency-dependent metasurfaces, etc
  - ✗ maths: non-smoothness of the solution across interfaces
- strategy: immersed interface method
  - ✓ modification of the numerical values used by the scheme
  - ✓ modified values based on geometry and jump conditions
  - ✓ preprocessing step



Touboul-Lombard-Bellis, JCP 2020

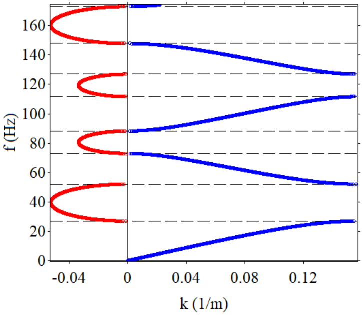
Part IV

Numerical examples

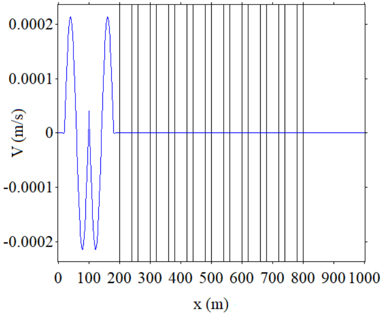
Periodic medium 1D

- periodic medium, cell of length h
  - ✓ Bloch-Floquet theory: $U(x+h) = U(x)\, e^{ikh}$, k solved on a single cell
  - ✓ $k \in \mathbb{C}$: existence of forbidden bands
- example: $f = 30$ Hz (**video**), $f = 60$ Hz (**video**)

dispersion diagrams



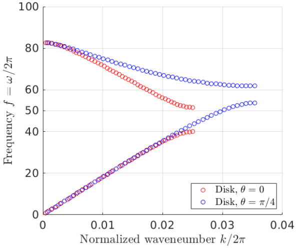
finite slab



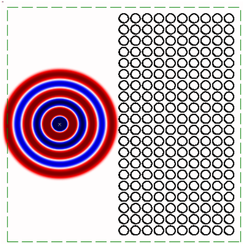
Periodic medium 2D

- fluid / fluid configuration
  - ✓ $f = 30$ Hz: no gap (**video**)
  - ✓ $f = 45$ Hz: gap at $\theta = 0$ (**video**)
  - ✓ $f = 58$ Hz: gap at $\theta = \pi/4$ (**video**)

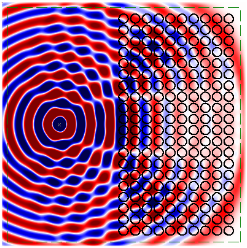
dispersion at $\theta = 0$ and $\theta = \pi/4$



$f = 45$ Hz



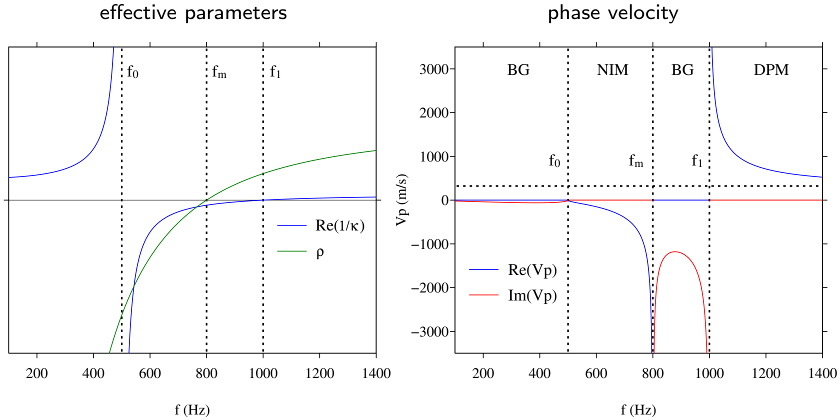
$f = 45$ Hz



Dispersive medium 1D (1/3)

- Drude / Drude-Lorentz effective model

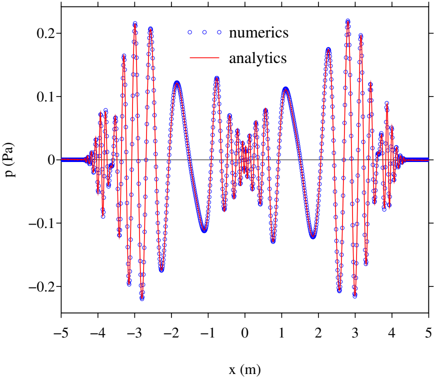
$$\hat{\rho}(\omega) = \rho_a \left(1 - \frac{\Omega_\rho^2}{\omega^2}\right) \qquad \hat{\kappa}^{-1}(\omega) = \kappa_a^{-1}\left(1 - \frac{\Omega_\kappa^2}{\omega^2 - \omega_\kappa^2}\right)$$



Dispersive medium 1D (2/3)

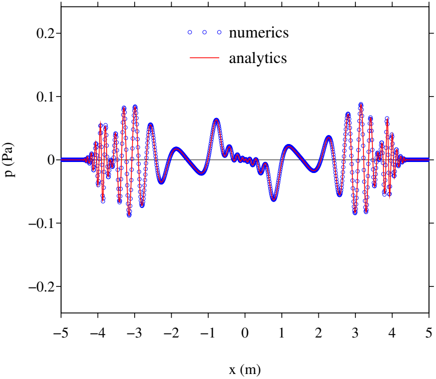
- inviscid case ($\gamma = 0$) or dissipative case ($\gamma = 400$)
  - ✓ comparison with semi-analytical solution (Fourier)
  - ✓ backward propagative waves at frequencies $\in [f_0, f_m]$ (NIM)
  - ✓ pure sinus: **video**

inviscid case ($\gamma = 0\,\mathrm{s}^{-1}$)



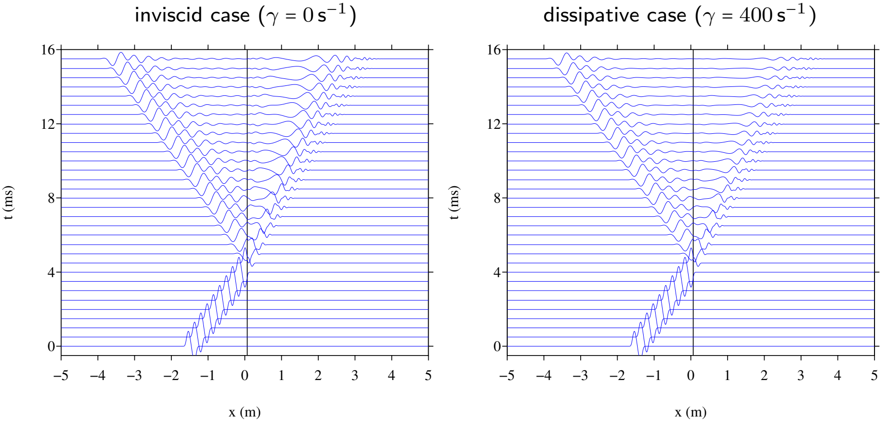
video

dissipative case ($\gamma = 400\,\mathrm{s}^{-1}$)



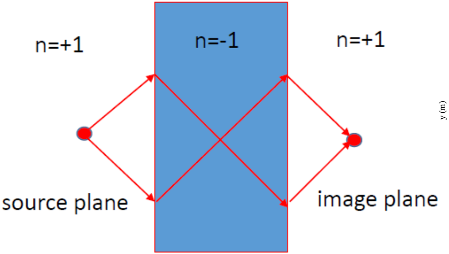
video

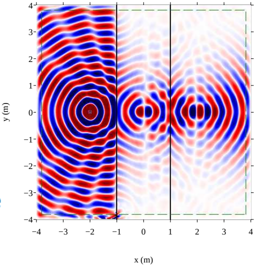
- interface fluid / metamaterial
  - ✓ dispersive transmitted wave



video **video**

- Pendry Lens
  - ✓ resonant medium in the slab



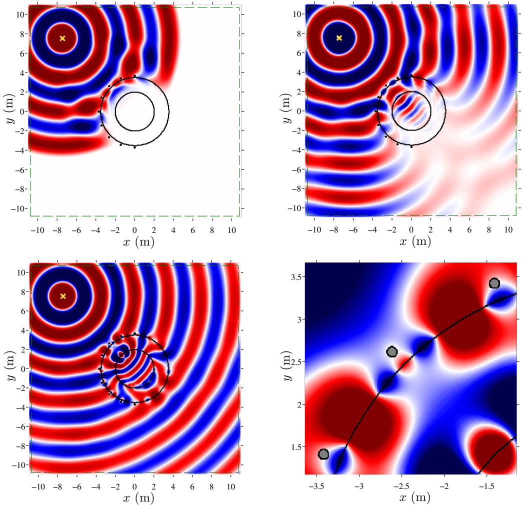


video

📚 Bellis-Lombard, Wave Motion 2019

- external cloak

Guenneau-Lombard-Bellis, APL (2021)

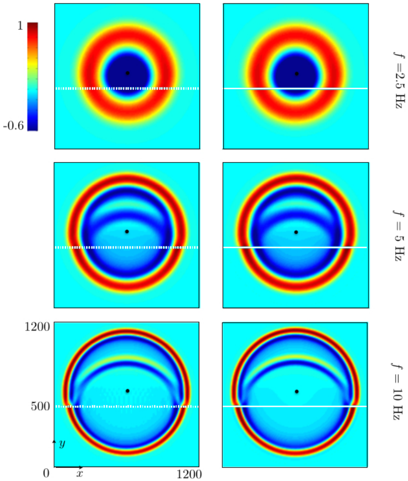


video **video**

- line of rigid scatterers

Lombard-Maurel-Marigo, JCP (2017)

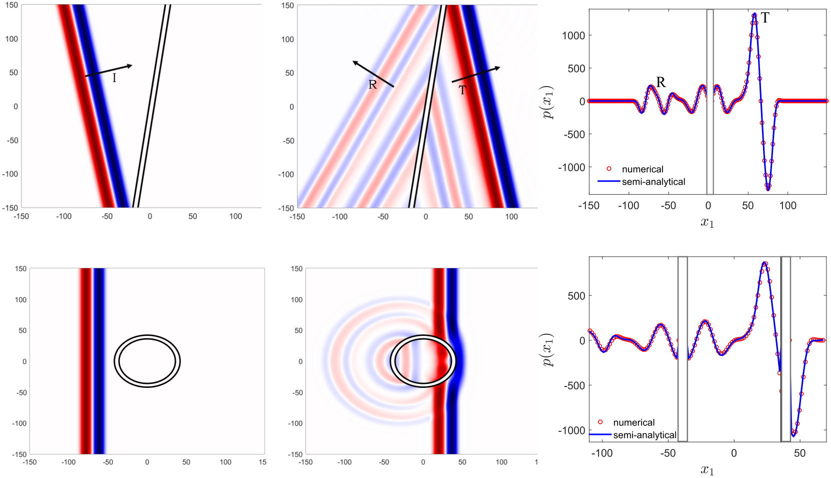
$$\begin{cases} [\![p]\!] = N\langle \partial_n p \rangle \\ [\![v_n]\!] = C_1 \langle \partial_n v_n \rangle + C_2 \langle \partial_\tau v_\tau \rangle \end{cases}$$



Interface homogenization: resonant case (1/2)

- resonant metasurface (Touboul-Lombard-Bellis, JCP 2020)

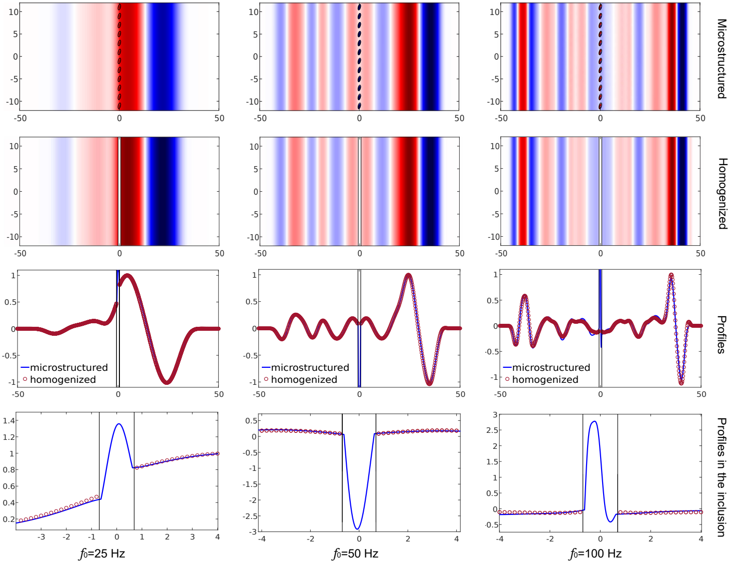
$$\begin{cases} [\![\hat{p}]\!]_a = \boldsymbol{B}.\langle \nabla \hat{p} \rangle_a \\ [\![\hat{\boldsymbol{v}}.\boldsymbol{n}]\!]_a = \boldsymbol{C} : \langle \nabla \hat{\boldsymbol{v}} \rangle_a + \mathscr{D}_\infty(\omega) \langle \text{div } \hat{\boldsymbol{v}} \rangle_a , \end{cases} \qquad \mathscr{D}_\infty(\omega) = \alpha_0 - \sum_{r \geq 1} {\alpha_r}^2 \frac{\omega^2}{\omega^2 - {\omega_r}^2}$$



Interface homogenization: resonant case (2/2)

- full field vs homogenization

Touboul-Pham-Maurel-Marigo-Lombard-Bellis, JElas 2020



Nonlinear media: acoustic solitons (1/2)

- guide connected with Helmholtz resonators (O. Richoux, LAUM)

Sugimoto, JFM 1995

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial}{\partial x}\left(a u + b\dfrac{(u)^2}{2}\right) = c\dfrac{\partial^{-1/2}}{\partial t^{-1/2}}\dfrac{\partial u}{\partial x} + d\dfrac{\partial^2 u}{\partial x^2} - e\dfrac{\partial p}{\partial t} \\ \dfrac{\partial^2 p}{\partial t^2} + f\dfrac{\partial^{3/2} p}{\partial t^{3/2}} + gp - m\dfrac{\partial^2 (p)^2}{\partial t^2} + n\left|\dfrac{\partial p}{\partial t}\right|\dfrac{\partial p}{\partial t} = hu \end{cases}$$

✓ nonlinear propagation: a, b

✓ oscillator: g

✓ coupling: e, h

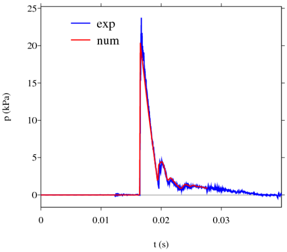
✓ attenuation: bulk d, fractional c and f, nonlinear m and n

Nonlinear media: acoustic solitons (2/2)

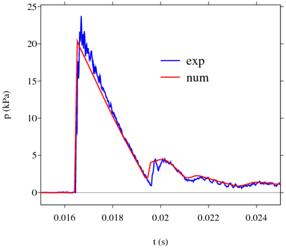
- comparisons experiences / simulations

📚 Richoux-Lombard-Mercier, Wave Motion 2015

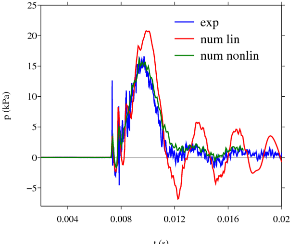
no resonators



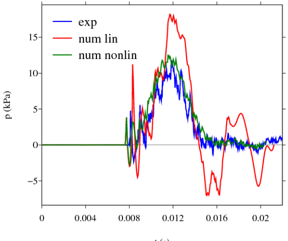
no resonators (zoom)



height $H = 7$ cm



height $H = 13$ cm



Part V

Conclusion

Take-home message

- time-domain studies:
  - ✓ possible even if frequency dependent parameters
  - ✓ advantages: nonlinearities, transients, etc
  - ✓ good formalism: hyperbolic models, conservative form, dedicated schemes
- bibliography:

  - R. LeVeque, Numerical Methods for Conservation Laws (1993)
  - M. Touboul, PhD Thesis (2021)
  - Author's webpage at LMA

Thanks for your attention!

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